International Conference of Physics (2016)

Ganaralization of Variational Principle

For unified description of Quantum and Classical Physics

Tomoi Koide (IF,UFRJ)

Pedagogical introduction,
Koide et. al., J. Phys. Conf. 626, 012055 (`15)

Variational Principle in Class. Mech.

$$I(x) = \int_{t_I}^{t_F} dt \left[\frac{m}{2} \left(\frac{dx(t)}{dt} \right)^2 - V(x(t)) \right]$$

OPTIMIZATION

Newton equation

Variational Principle in Quan. Mech.

$$I(\phi,\phi^*) = \int_{V_I}^{V_F} d^4x\, \phi^* \left[i\hbar\partial_t - H \right] \phi \xrightarrow{\text{OPTIMIZATION}} \text{Schrödinger equation}$$

Note that
- Variation of WF, not trajectory
- $I(\phi,\phi^*) \neq T - V$

Path Integral Approach

$$\langle a|b\rangle = \int_a^b \int [Dx] \exp(iI) \Rightarrow$$

All paths contribute!

Quantum path still satisfies the law of optimization.

↓

For this, we need to extend the formulation of the variational method.

HOW ?

A

B

Optimized path ?

A

B

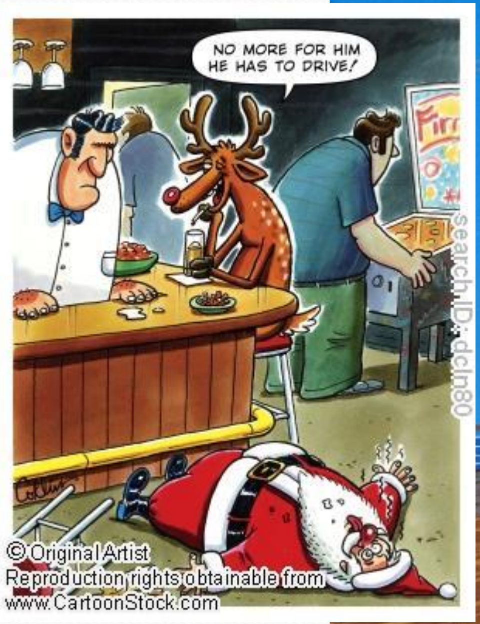
Optimized path ?

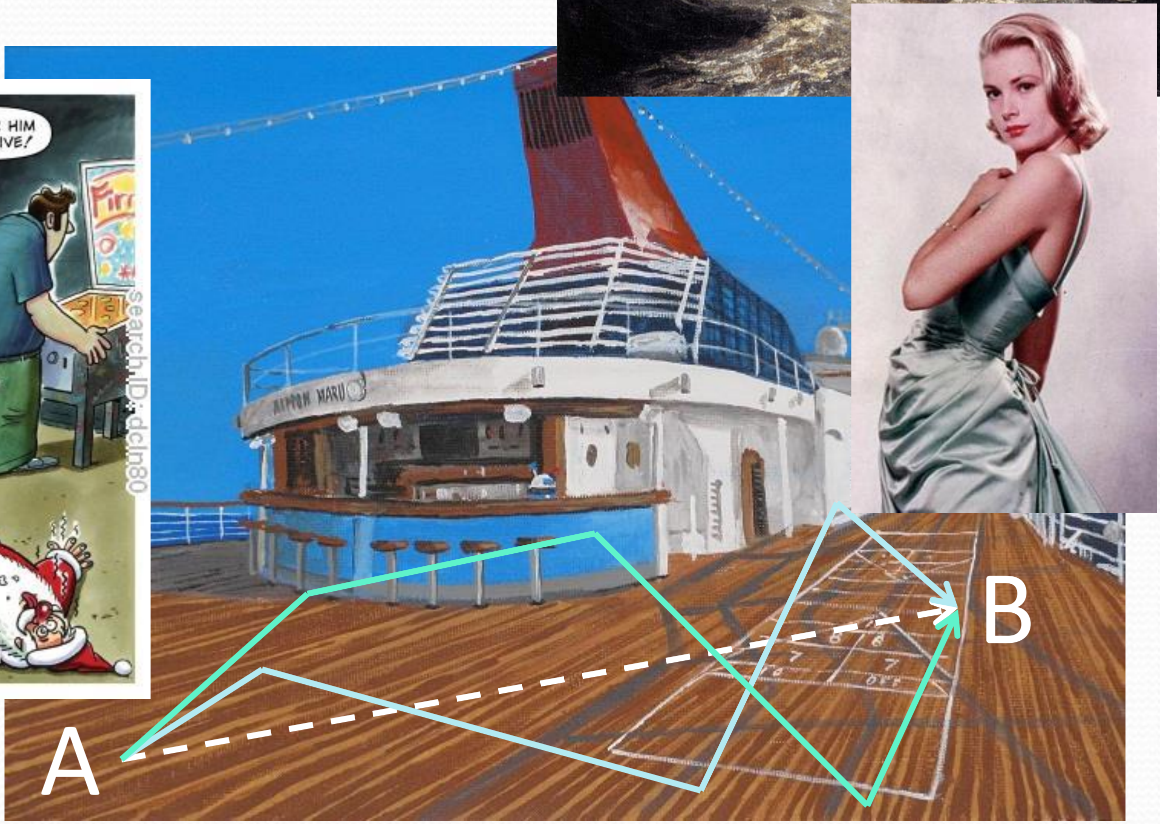

Queen Victoria


B
A
Optimized path !

We cannot follow the optimized path!!

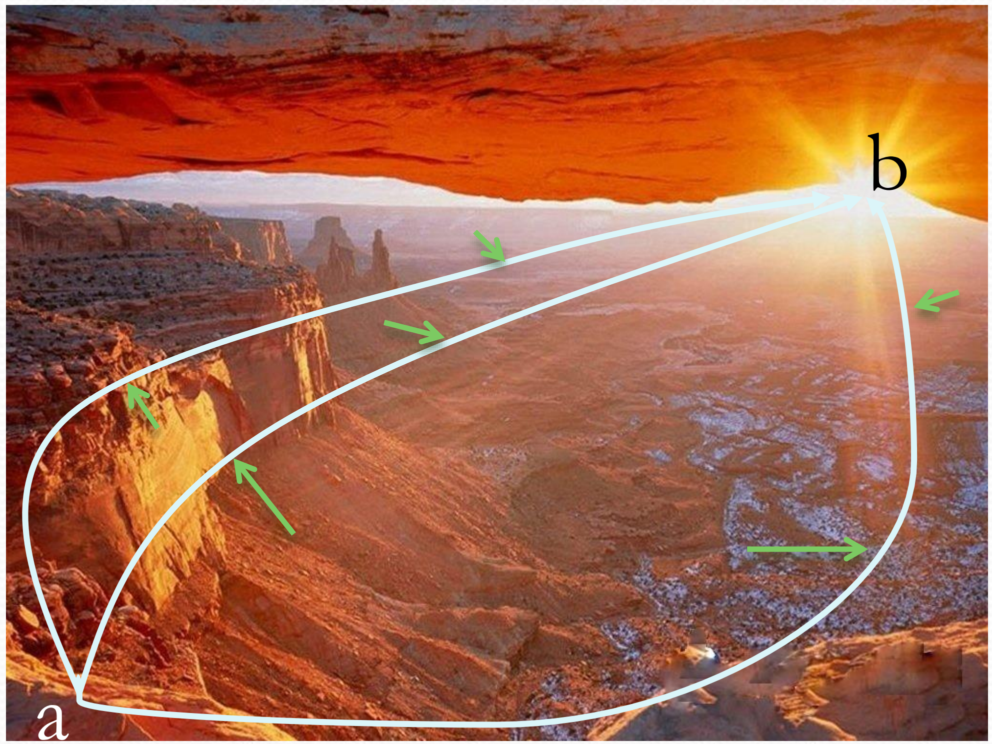
1. Zig-zag by fluctuation
2. Each times, different paths





Optimal control

Effect of variables which we cannot control.



$$I(x) = \int_{t_I}^{t_F} dt \left[\frac{m}{2}\left(\frac{dx(t)}{dt}\right)^2 - V(x(t)) \right] \Longrightarrow$$ Newton equation

- Yasue, J. Funct. Anal. 41 327 (1981)
- Nelson, Quantum Fluctuations (Princeton, NJ: Princeton University Press, 1985)
- Guerra&Morato, Phys. Rev. D27 1774 (1983)
- Pavon, J. Math. Phys. 36 6774 (1995)
- Nagasawa, Stochastic Process in Quantum Physics (Bassel:Birkhaeuser, 2000)
- Cresson and Darses , J. Math. Phys. 48 072703 (2007)
- Holm, arXiv:1410.8311 [math-ph]
- Chen,Cruzeiro&Ratiu, arXiv:1506.05024 [math-ph]

⋮

Stochastic Variational method

is one approach to calculate optimization

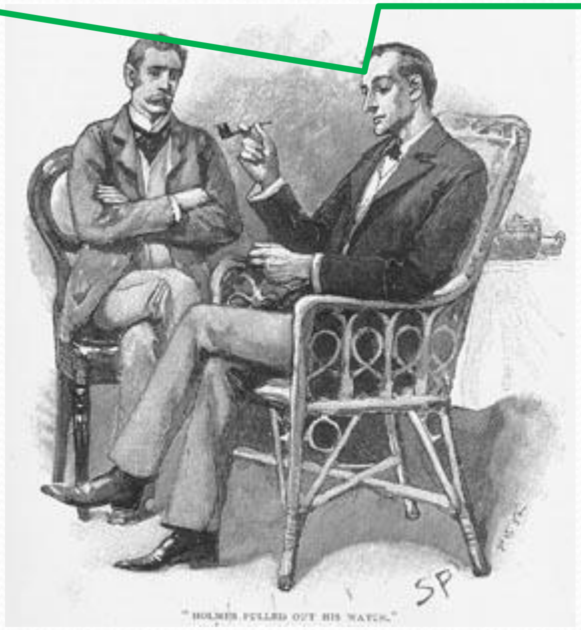
including such a fluctuation.

Formulation of SVM

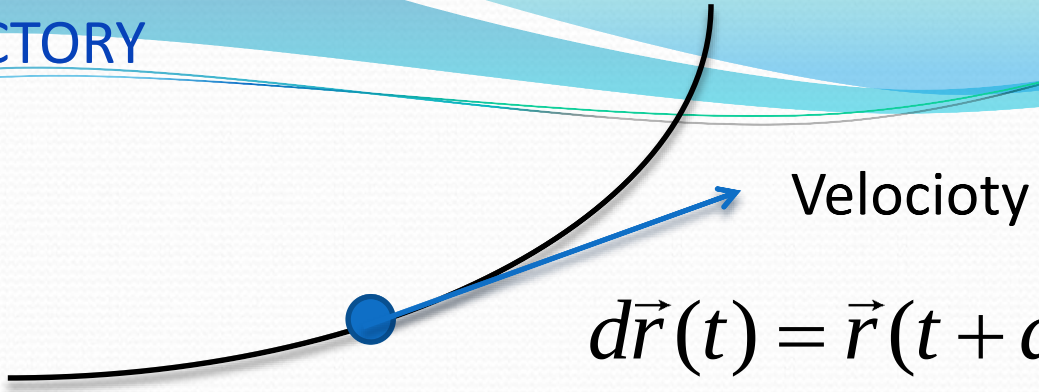
KEY POINT

Definition of velocity!

5 important steps



CLASSICAL TRAJECTORY

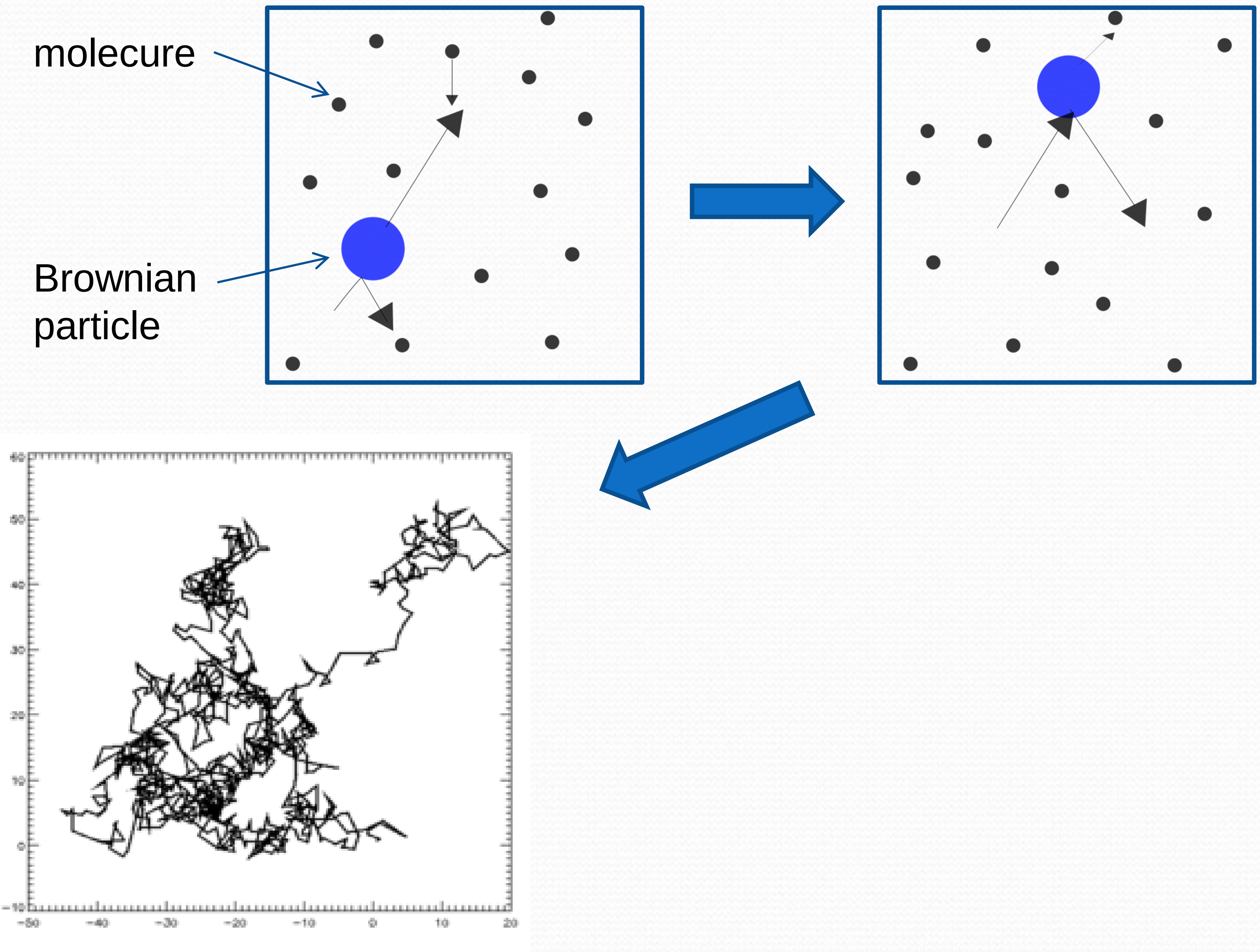


$$d\vec{r}(t) = \vec{r}(t+dt) - \vec{r}(t)$$

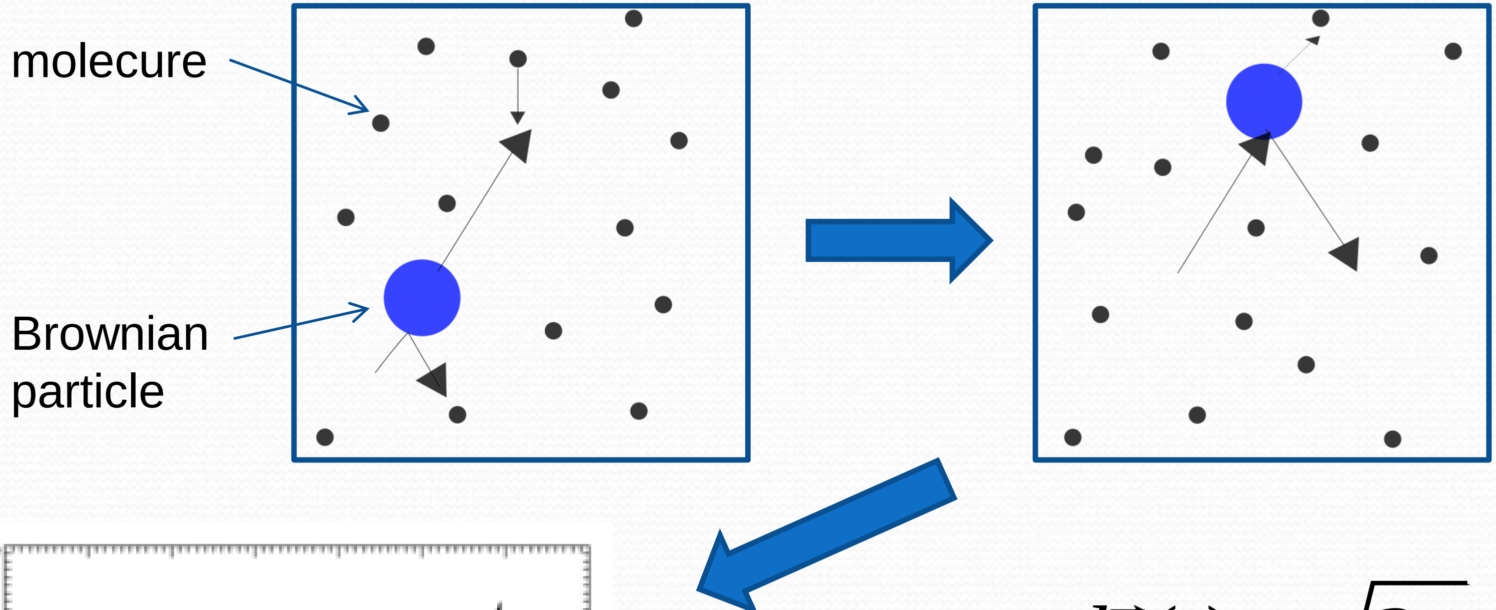
$$= \vec{u}dt$$

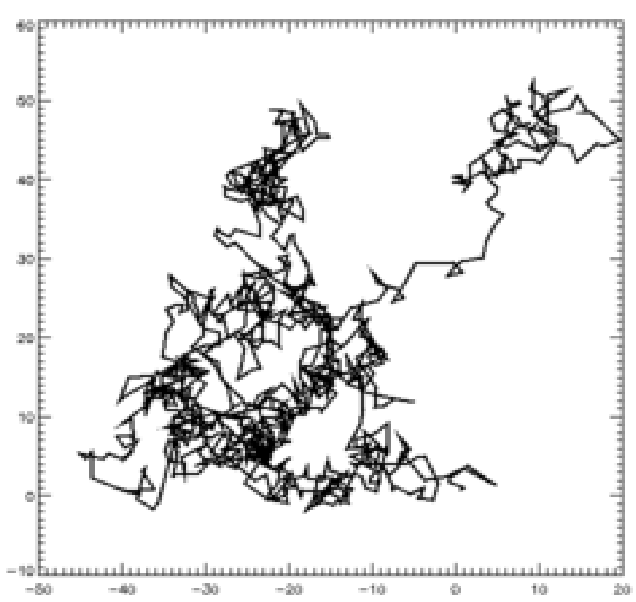
Zig-zag motion ?

Zig-zag in Brownian motion



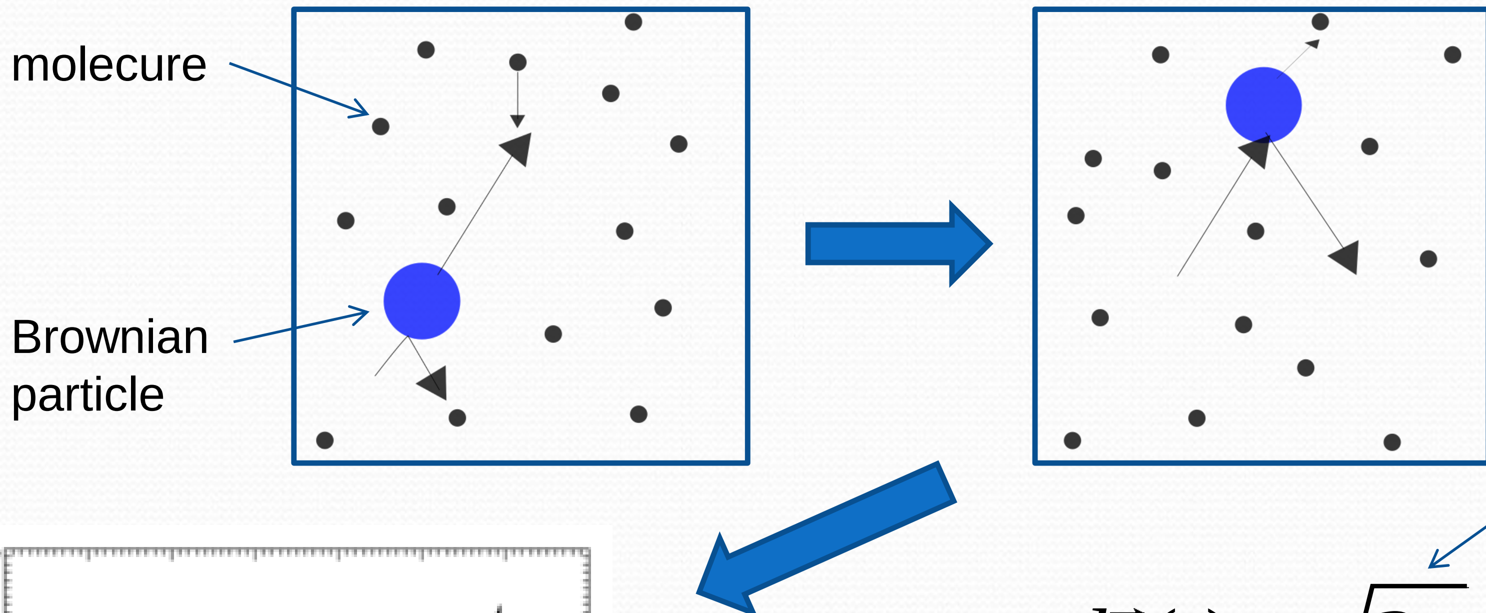
Zig-zag in Brownian motion

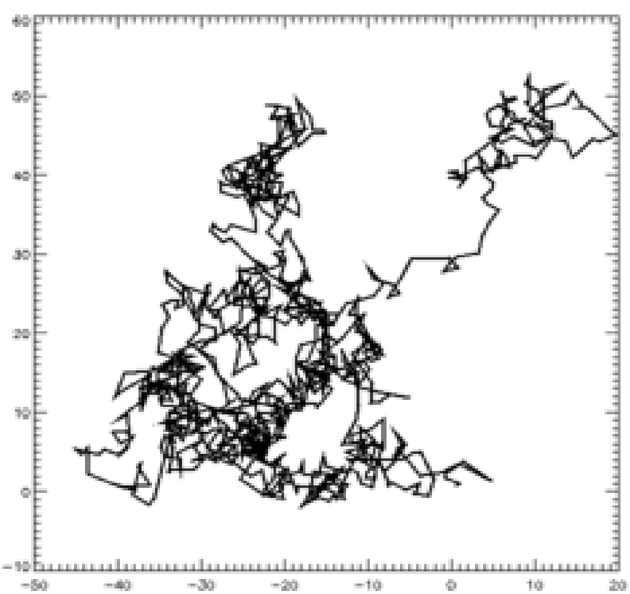




$$d\vec{r}(t) = \sqrt{2\nu} \cdot d\vec{W}(t)$$

Zig-zag in Brownian motion





Noise intensity

$$d\vec{r}(t) = \sqrt{2\nu} \cdot d\vec{W}(t)$$

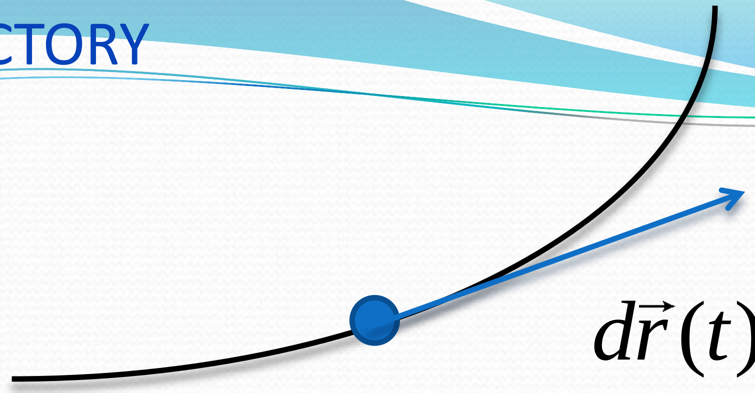
Gaussian white noise
(Wiener process)

$$E[d\vec{W}] = 0 \qquad E[(dW_i)^2] = dt$$

Direction is random

Mean magnitude is $\sqrt{dt}$

CLASSICAL TRAJECTORY

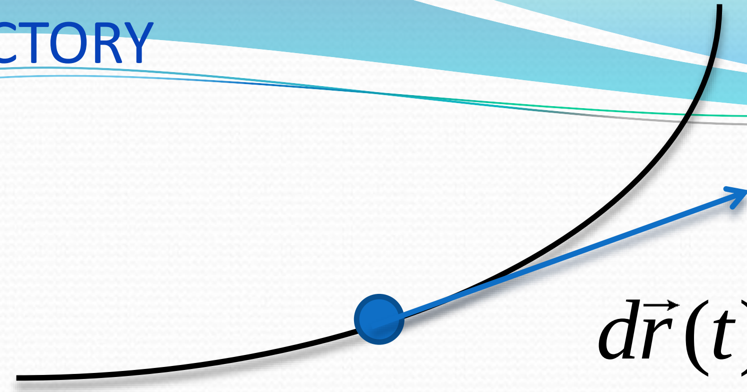


$$d\vec{r}(t) = \vec{r}(t+dt) - \vec{r}(t)$$

$$= \vec{u}dt$$

Zig-zag motion ?

CLASSICAL TRAJECTORY

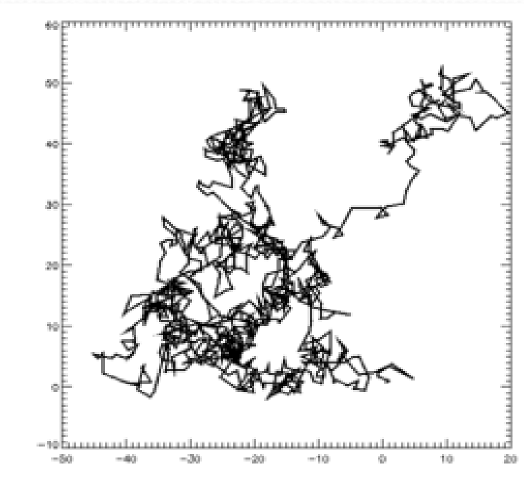


Velocioty

$$d\vec{r}(t) = \vec{r}(t+dt) - \vec{r}(t)$$

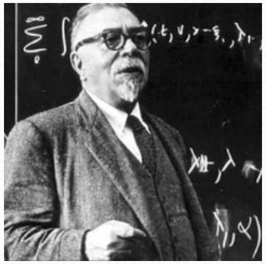
$$= \vec{u}dt$$

STOCHASTIC TRAJECTORY



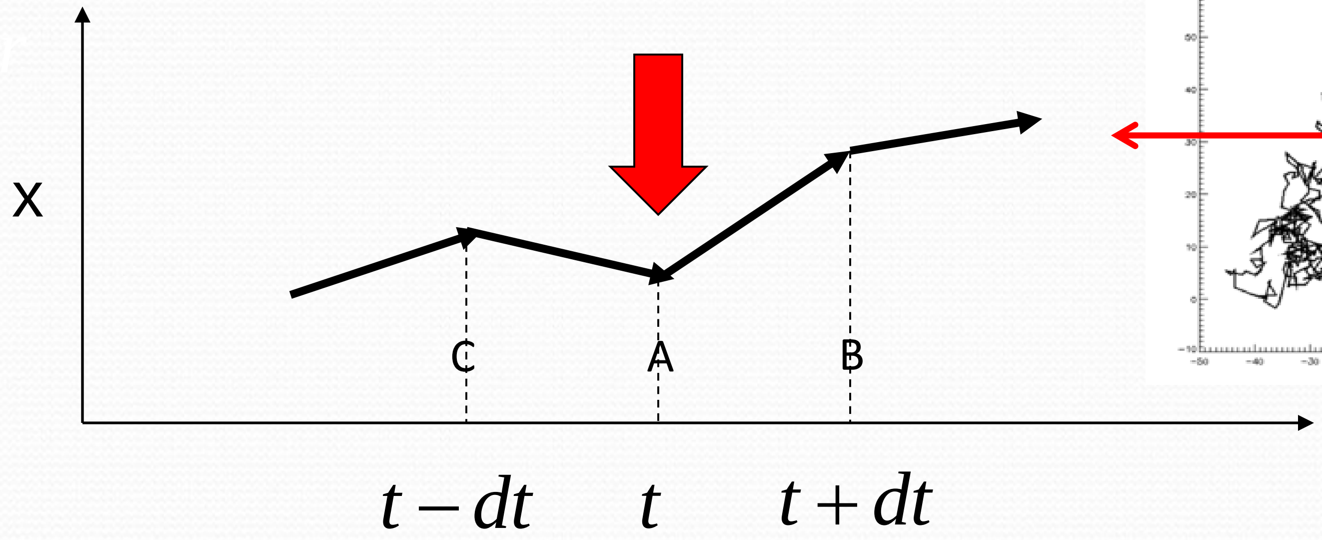
$$d\vec{r}(t) = \vec{u}(\vec{r}(t),t)\underline{dt} + \sqrt{2\nu}\cdot d\vec{W}(t)$$

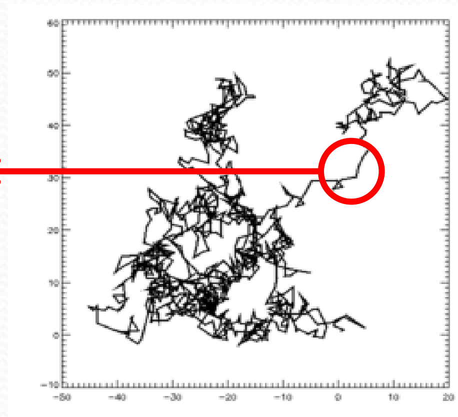
$$> 0$$



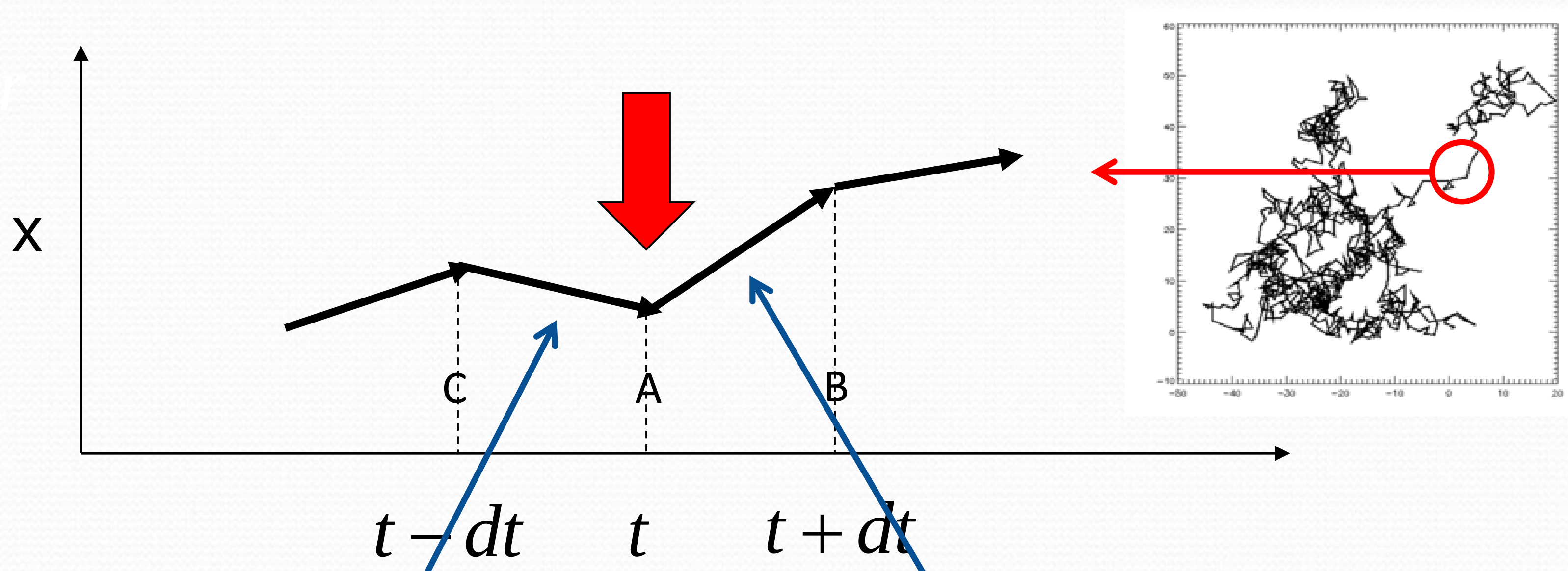
$$E[d\vec{W}] = 0 \qquad E[(dW_i)^2] = dt$$

How define the velocity at A?





How define the velocity at A?



$$\lim_{dt\to 0+}\frac{\vec{r}(t+dt)-\vec{r}(t)}{dt}$$

$$\lim_{dt\to 0-}\frac{\vec{r}(t+dt)-\vec{r}(t)}{dt}$$

Bernstein Process

Forward stochastic differential equation $\left(dt > 0\right)$

$$d\vec{r}(t) = \vec{u}(\vec{r}(t),t)dt + \sqrt{2\nu} \cdot d\vec{W}(t) \qquad E\left[(dW_i(t))^2\right] = dt$$

We employ the variational procedure to determine these unknown functions.

Bernstein Process

Forward stochastic differential equation $(dt > 0)$

$$d\vec{r}(t) = \vec{u}(\vec{r}(t), t)dt + \sqrt{2\nu} \cdot d\vec{W}(t) \qquad E\left[(dW_i(t))^2\right] = dt$$

Backward stochastic differential equation $(dt < 0)$

$$d\vec{r}(t) = \vec{\tilde{u}}(\vec{r}(t), t)dt + \sqrt{2\nu} \cdot d\vec{\tilde{W}}(t) \qquad E\left[(d\tilde{W}_i(t))^2\right] = -dt$$

We employ the variational procedure to determine these unknown functions.

Consistency Condition

Probability density $\rho(\vec{x},t) = E\left[\delta(\vec{x}-\vec{r}(t))\right]$

The Fokker-Plank equation (forward)

$$\partial_t \rho = -\nabla\left(\vec{u} - \nu\nabla\right)\rho$$

The Fokker-Plank equation (backward)

$$\partial_t \rho = -\nabla\left(\vec{\tilde{u}} + \nu\nabla\right)\rho$$

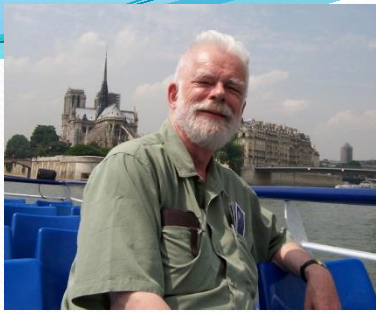
These two should be equivalent

2

$$\vec{u} = \vec{\tilde{u}} + 2\nu\nabla \ln \rho$$

Also related to Bayesian statistics
Caticha, JPA44, 225303(`11)

Time Drivative Operations



Because of the two different definitions of velocities, we can introduce the two different time derivatives.

(Nelson)

3

Mean forward derivative $\quad D\vec{r}\left(\simeq \lim_{dt\to 0+}\frac{d\vec{r}}{dt}\right)=\vec{u}$

Mean backward derivative $\quad \tilde{D}\vec{r}\left(\simeq \lim_{dt\to 0-}\frac{d\vec{r}}{dt}\right)=\vec{\tilde{u}}$

See, Koide, Kodama&Tsushima, JP Conf. 626, 012055 (`15)

Zheng&Meyer, S`eminarie de Probabilit`es XVIII, Lecture Notes in Mathematics Vol 1059, P223, (Springer-Verlag, Berlin, 1984).

Partial Integration Formula

CLASSICAL

$$\int_a^b dt \frac{dX}{dt} \cdot Y = \left[X(b)Y(b) - X(a)Y(a) \right] - \int_a^b dt X \cdot \frac{dY}{dt}$$

STOCHASTIC

4

$$\int_a^b dt E\left[(DX) \cdot Y \right]$$

$$= E\left[X(b)Y(b) - X(a)Y(a) \right] - \int_a^b dt E\left[X \cdot (\tilde{D}Y) \right]$$

See, Koide, Kodama&Tsushima, JP Conf. 626, 012055 (`15)

Ito Formula (Ito's lemma)

This is a kind of Tayler expansion for stochastic variables.

Taylor

$$d\vec{r} = \vec{u}dt$$

$$df(\vec{r}(t),t) = dt\left[\partial_t + u\cdot\nabla\right] f(\vec{r}(t),t) + O(dt^2)$$

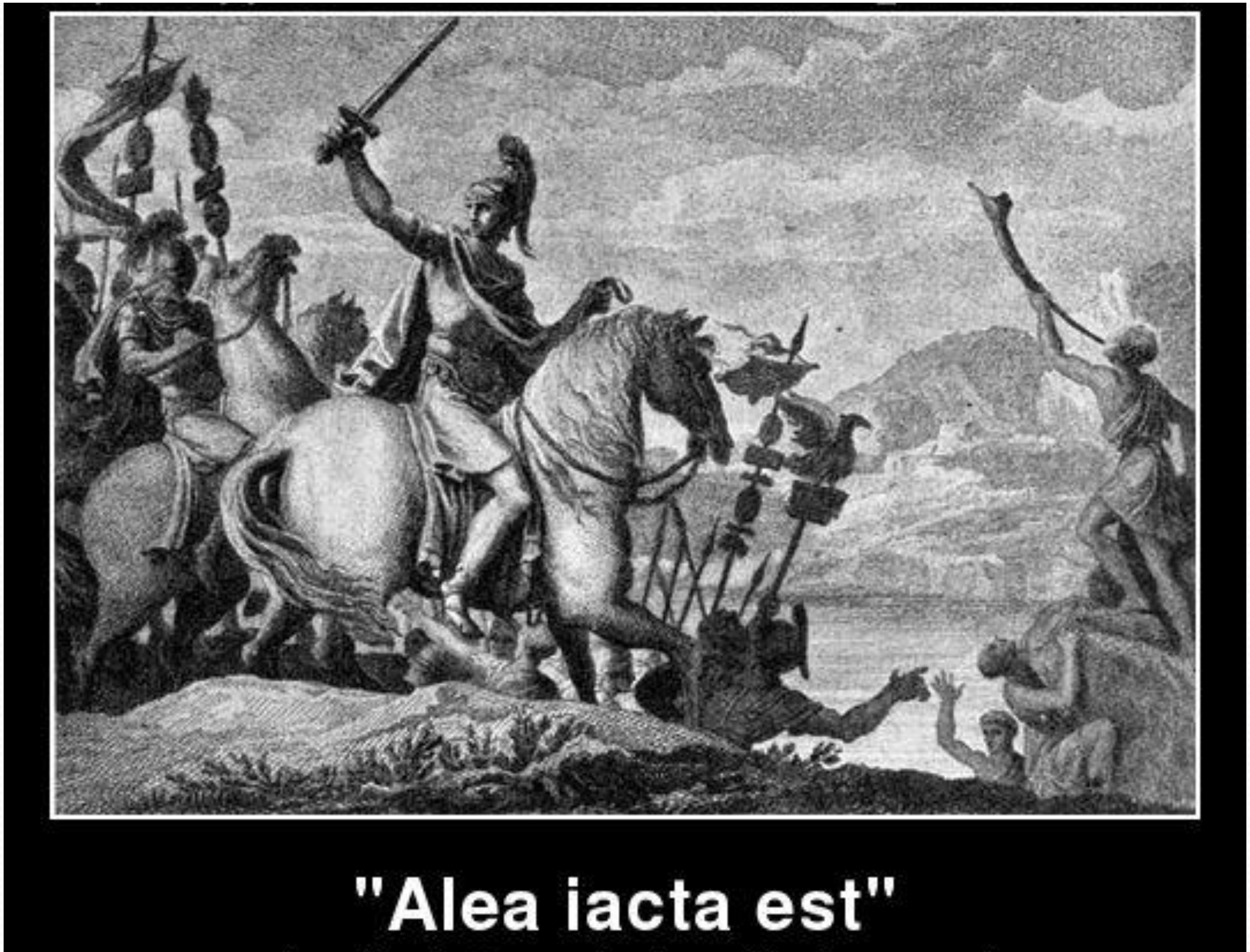
5

Ito

$$d\vec{r} = \vec{u}dt + \sqrt{2\nu}\,d\vec{W}$$

$$df(\vec{r}(t),t) = dt\left[\partial_t + u\cdot\nabla + \nu\nabla^2\right] f(r(t),t) + \sqrt{2\nu}\nabla f\cdot dW$$
$$+O(dt^2)$$

Let's apply!!



Stochastic Representation of Action

Classical action
$$I_{cla} = \int_a^b dt \left(\frac{m}{2} \left(\frac{d\vec{r}(t)}{dt} \right)^2 - V(\vec{r}(t)) \right)$$

For example

$$\left(\frac{d\vec{r}}{dt} \right)^2 \Rightarrow \begin{cases} 1) \quad D\vec{r} \cdot D\vec{r} \\ 2) \quad \tilde{D}\vec{r} \cdot \tilde{D}\vec{r} \\ 3) \quad \dfrac{D\vec{r} \cdot D\vec{r} + \tilde{D}\vec{r} \cdot \tilde{D}\vec{r}}{2} \end{cases}$$

We consider 3)

Stochastic action
$$I_{sto} = \int_a^b dt E\left[\frac{m}{2} \frac{(D\vec{r})^2 + (\tilde{D}\vec{r})^2}{2} - V(\vec{r}) \right]$$

Stochastic Variation for Kinetic Term

$$\vec{r} \to \vec{r} + \delta\overleftarrow{r} \qquad \delta\overleftarrow{r}(a) = \delta\overleftarrow{r}(b) = 0$$

$$\delta\int_a^b dt\,\frac{m}{2}E\left[(D\vec{r})\cdot(D\vec{r})\right] = m\int_a^b dtE\left[(D\vec{r})\cdot(D\delta\vec{r})\right]$$

$$= m\int_a^b dtE\left[\vec{u}\cdot(D\delta\vec{r})\right]$$

$$= -m\int_a^b dtE\left[\tilde{D}\vec{u}\cdot\delta\vec{r}\right]$$

Ito formula $\qquad \tilde{D}\vec{u}(\vec{r},t) = \lim_{dt\to 0_-}\dfrac{d\vec{u}(\vec{r},t)}{dt} = \left(\partial_t + \vec{\tilde{u}}\cdot\nabla - \nu\Delta\right)\vec{u}$

Variation of Action

$$\delta I = 0 \Rightarrow \boxed{\left(\partial_t + \vec{u}_m \cdot \nabla\right)\vec{u}_m - 2\nu^2 \nabla\left(\rho^{-1/2}\nabla^2\sqrt{\rho}\right) = -\frac{1}{m}\nabla V(\vec{r})}$$

$$\vec{u}_m = (\vec{u} + \vec{\tilde{u}})/2$$

Variation of Action

$$\delta I = 0 \Rightarrow \boxed{\left(\partial_t + \vec{u}_m \cdot \nabla\right)\vec{u}_m - 2\nu^2 \nabla\left(\rho^{-1/2}\nabla^2\sqrt{\rho}\right) = -\frac{1}{m}\nabla V(\vec{r})}$$

$$\vec{u}_m = (\vec{u} + \vec{\tilde{u}})/2$$

when $\nu = 0 \Rightarrow$

Variation of Action

$$\delta I = 0 \Rightarrow \boxed{\underline{\left(\partial_t + \vec{u}_m \cdot \nabla\right)} \vec{u}_m - 2\nu^2 \nabla\left(\rho^{-1/2} \nabla^2 \sqrt{\rho}\right) = -\frac{1}{m} \nabla V(\vec{r})}$$

$$\vec{u}_m = (\vec{u} + \vec{\tilde{u}})/2$$

$$\left(\partial_t + \vec{u}_m \cdot \nabla\right) = \frac{d}{dt}$$

when $\nu = 0$ $\Rightarrow$ The Newton equation

$$\frac{d}{dt} \vec{u}_m(\vec{r}(t), t) = -\frac{1}{m} \nabla V(\vec{r}(t))$$

Variation of Action

$$\delta I = 0 \Rightarrow \boxed{\left(\partial_t + \vec{u}_m \cdot \nabla\right)\vec{u}_m - 2\nu^2 \nabla\left(\rho^{-1/2}\nabla^2\sqrt{\rho}\right) = -\frac{1}{m}\nabla V(\vec{r})}$$

$$\vec{u}_m = (\vec{u} + \vec{\tilde{u}})/2$$

$$\left(\partial_t + \vec{u}_m \cdot \nabla\right) = \frac{d}{dt}$$

when $\nu = 0$ $\Rightarrow$ The Newton equation

$$\frac{d}{dt}\vec{u}_m(\vec{r}(t),t) = -\frac{1}{m}\nabla V(\vec{r}(t))$$

The dynamics of ρ is given by the FP equation.

$$\partial_t \rho = -\nabla\left(\vec{u} - \nu\nabla\right)\rho = -\nabla\left(\rho\vec{u}_m\right)$$

Derivation of Schrödinger Equation

Introduction of phase $\nabla \vartheta = \vec{u}_m / (2\nu)$

Eq. of variation

$$\partial_t \vartheta + \nu \left(\nabla \vartheta \right)^2 - \nu \left(\rho^{-1/2} \nabla^2 \sqrt{\rho} \right) + \frac{1}{m} \nabla V = 0$$

Derivation of Schrödinger Equation

Introduction of phase $\nabla \vartheta = \vec{u}_m / (2\nu)$

Eq. of variation $\longrightarrow$ $\partial_t \vartheta + \nu (\nabla \vartheta)^2 - \nu \left(\rho^{-1/2} \nabla^2 \sqrt{\rho} \right) + \frac{1}{m} \nabla V = 0$

Introduction of wave function $\varphi \equiv \sqrt{\rho} e^{i\vartheta}$

Yasue, JFA 41, 327 (`81)

$$i\partial_t \varphi = \left[-\nu \Delta + \frac{1}{2\nu m} V \right] \varphi \xrightarrow{\nu = \frac{\hbar}{2m}} i\hbar \partial_t \varphi = \left[-\frac{\hbar^2}{2m} \Delta + V \right] \varphi$$

The Schrödinger equation

Lagrangian

Optimization
in macro. scale

Optimization
in micro. scale

Newton equation

Schrödinger equation

NOISE

GLASSES

Canonical Quantization and SVM

$$L(r,\dot r)$$

optimization

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r}-\frac{\partial L}{\partial r}=0$$

LEGENDRE TRANSFORM

$$H(r,p)$$

$$p=\partial L/\partial \dot r$$

optimization

$$\dot r=\{r,H\}_{PB}$$

$$\dot p=\{p,H\}_{PB}$$

Canonical Quantization and SVM

$$L(r,\dot{r})$$

optimization

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

LEGENDRE TRANSFORM

$$H(r,p)$$

$$p = \partial L / \partial \dot{r}$$

optimization

canonical quantization

$$\dot{r} = \{r, H\}_{PB} \rightarrow \frac{1}{i\hbar}[\hat{r}, \hat{H}]$$

$$\dot{p} = \{p, H\}_{PB} \rightarrow \frac{1}{i\hbar}[\hat{p}, \hat{H}]$$

Canonical Quantization and SVM

$L(r,\dot{r})$

optimization

SVM

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}}-\frac{\partial L}{\partial r}=0$$

$$\tilde{D}\frac{\partial L}{\partial (Dr)}+D\frac{\partial L}{\partial (\tilde{D}r)}-\frac{\partial L}{\partial r}=0$$

LEGENDRE TRANSFORM

$H(r,p)$

$p=\partial L/\partial \dot{r}$

optimization

canonical quantization

$$\dot{r}=\left\{r,H\right\}_{PB}$$
$$\rightarrow\frac{1}{i\hbar}[\hat{r},\hat{H}]$$
$$\dot{p}=\left\{p,H\right\}_{PB}$$
$$\rightarrow\frac{1}{i\hbar}[\hat{p},\hat{H}]$$

Noether Theorem

Misawa, J. Math. Phys. 29 2178 (`88)

Invariance for spatial translation

$$\vec{r}(t) \longrightarrow \vec{r}(t)+\vec{A}$$

The change
of the action

$$\Rightarrow \quad \delta I = \int_{t_i}^{t_f} dt E\left[L(\vec{r}+\vec{A}, D\vec{r}, \tilde{D}\vec{r})\right] - \int_{t_i}^{t_f} dt E\left[L(\vec{r}, D\vec{r}, \tilde{D}\vec{r})\right]$$

$$= \int_{t_i}^{t_f} dt \frac{d}{dt} E\left[\frac{m}{2} D\vec{r} + \frac{m}{2}\tilde{D}\vec{r}\right] \cdot \vec{A}$$

Misawa, J. Math. Phys. 29 2178 (`88)

Invariance for spatial translation

$$\vec{r}(t) \longrightarrow \vec{r}(t) + \vec{A}$$

The change of the action

$$\Longrightarrow \quad \delta I = \int_{t_i}^{t_f} dt E\left[L(\vec{r}+\vec{A}, D\vec{r}, \tilde{D}\vec{r}) \right] - \int_{t_i}^{t_f} dt E\left[L(\vec{r}, D\vec{r}, \tilde{D}\vec{r}) \right]$$

$$= \int_{t_i}^{t_f} dt \frac{d}{dt} E\left[\frac{m}{2} D\vec{r} + \frac{m}{2} \tilde{D}\vec{r} \right] \cdot \vec{A}$$

If the action is invariant for the spatial translation,

conserved

momentum operator!

$$\frac{m}{2} E\left[D\vec{r} + \tilde{D}\vec{r} \right] = \int d^3x \, \varphi^*(\vec{x},t)(-i\hbar\partial_x)\varphi(\vec{x},t)$$

Comment : we do not need the normal ordering product. Koide&Kodama, PTEP 093A03 ('15)

Many particle systems

Classical Many-Body Dynamics

coase-grainings

Positions and velocities of all particles

Mass density and velocity of fluid

classical variation ↓ SVM

N-body Newton's eq.

classical variation ↓ SVM

Ideal fluid eq. (Euler)

Loffredo&Morato, JPA40,8709(`07), Koide, PLA379,2007(`15)

Classical Many-Body Dynamics

coase-grainings

Positions and velocities of all particles

Mass density and velocity of fluid

classical variation ↓ N-body Newton's eq.

SVM ↓ N-body Scrödinger eq.

classical variation ↓ Ideal fluid eq. (Euler)

SVM ↓ Gross -Ptaevskii eq.

Loffredo&Morato, JPA40,8709(`07), Koide, PLA379,2007(`15)

Concluding Remarks

- SVM is a useful method of for quantization of non-relativistic particles and bosonic fileds.
- SVM is applicable as a method for coase-grainings of micro. dynamics (Navier-Stokes-Fourier, Gross-Pitaevskii).
- The stochastic Noether theorem
- The uncertainty relations
- Quantum-Classical hybrids

SVM

Analytical Mechanics

stochastic particle

classical field

stochasic field

classical particle

Noether theorem

curved space-time

dissipative systems

rigid body

fermion field

brachistochrone curve

anomaly

canonical transform

Stochastic Analytical Mechanics

SVM

Analytical Mechanics

stochastic particle

classical field

stochasic field

classical particle

Noether theorem

curved space-time

dissipative systems

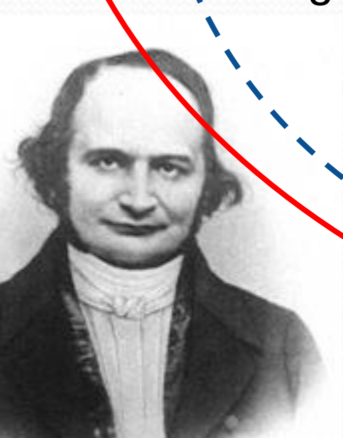
rigid body

fermion field

brachistochrone curve

anomaly

canonical transform



Pedagogical introduction,

Koide et. al.,
J. Phys. Conf. 626, 012055 (`15)

7th International Workshop DICE2014 Spacetime – Matter – Quantum Mechanics IOP Publishing
Journal of Physics: Conference Series **626** (2015) 012055 doi:10.1088/1742-6596/626/1/012055

Unified description of classical and quantum behaviours in a variational principle

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Abstract. We give a pedagogical introduction of the stochastic variational method and show that this generalized variational principle describes classical and quantum mechanics in a unified way.

1. Introduction

Variational approach conceptually plays a fundamental role in elucidating the structure of classical mechanics, clarifying the origin of dynamics and the relation between symmetries and conservation laws. In classical mechanics, the optimized function is characterized by Lagrangian, defined as $T - V$ with T and V being a kinetic and a potential terms, respectively.

We can still argue the variational principle in quantum mechanics, but the Lagrangian does not have any more the form of $T - V$, instead it is given by $\psi^*(i\hbar\partial_t - \hat{H})\psi$, where $\hat{H}$ is a Hamiltonian operator and ψ is a wave function. Therefore, at first glance, any clear or direct correspondence between classical and quantum mechanics does not seem to exist in the variational point of view, but it does exist. If we extend the idea of the variation to stochastic variable, the variational principle describes classical and quantum behaviors in a unified way.

This method is called stochastic variational method (SVM) and firstly proposed by Yasue [1, 2, 3, 4, 5] so as to reformulate Nelson's stochastic quantization [6, 7]. This framework is, however, based on special techniques attributed to stochastic calculus which is not familiar to physicists. In this paper, we give a pedagogical introduction of SVM in a self-contained manner, showing the unified description of classical and quantum mechanics. As another review, see, for example, Ref. [8].


ありがとう!
Köszönöm!
Gracias!
Merci!
OBRIGADO!
Grazie!
Danke!
Thank you!
謝謝!
Спасибо!


PROCURA-
SE

Back Up

Many particle systems

Classical Many-Body Dynamics

coase-grainings

Positions and velocities of all particles

Mass density and velocity of fluid

Classical Many-Body Dynamics

coase-grainings

Positions and velocities of all particles

Mass density and velocity of fluid

classical variation

N-body Newton's eq.

classical variation

Ideal fluid eq. (Euler)

Classical Many-Body Dynamics

coase-grainings →

Positions and velocities of all particles	Mass density and velocity of fluid
classical variation ↓ N-body Newton's eq.	classical variation ↓ Ideal fluid eq. (Euler)
SVM ↓ N-body Scrödinger eq.	SVM ↓ ?

Loffredo&Morato, JPA40,8709(`07), Koide, PLA379,2007(`15)

Classical variation of fluid

Action of (ideal) fluid

$$I(\rho_M,\vec{v}) = \int_{t_I}^{t_F} dt \int d^3x \left[\frac{\rho_M(\vec{x},t)}{2} \vec{v}^2(\vec{x},t) - \varepsilon(\rho_M) \right]$$

Internal energy density

Classical variation

Classical variation of fluid

Action of (ideal) fluid

Fluid velocity

$$I(\rho_M,\vec{v})=\int_{t_I}^{t_F}dt\int d^3x\left[\frac{\rho_M(\vec{x},t)}{2}\vec{v}^2(\vec{x},t)-\varepsilon\left(\rho_M\right)\right]$$

Mass density

Internal energy density

Classical variation

Classical variation of fluid

Action of (ideal) fluid

Fluid velocity

$$I(\rho_M,\vec{v}) = \int_{t_I}^{t_F} dt \int d^3x \left[\frac{\rho_M(\vec{x},t)}{2} \vec{v}^2(\vec{x},t) - \varepsilon(\rho_M) \right]$$

Mass density

Internal energy density

Classical variation

Euler equation

$$\left(\partial_t + \vec{v}\cdot\nabla\right)\vec{v} = -\frac{1}{\rho_M}\nabla P \qquad P = -\frac{d}{d(1/\rho_M)}\left(\frac{\varepsilon}{\rho_M}\right)$$

Pressure

$$TdS = dE - PdV \xrightarrow[\text{adiabatic}(dS=0)]{} P = (dE/dV)_S$$

Application of SVM

Applying SVM to the same action of (ideal) fluid,

Noise intensity

Koide&Kodama, JPA45, 255204 (`12)

$$i\partial_t \varphi = \left[-\nu \Delta + \frac{1}{2\nu} \frac{d\varepsilon}{d\rho_M} \right] \varphi$$

$$\nabla \vartheta = \frac{1}{2\nu} \vec{u}_m$$

$$\varphi \equiv \sqrt{\rho} e^{i\vartheta}$$

When we choose

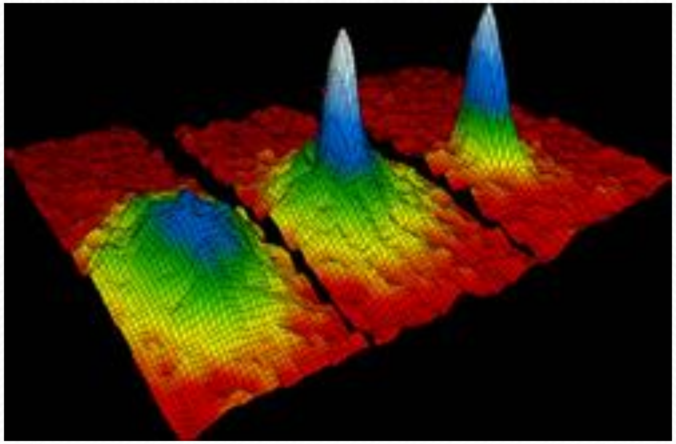
$$\underbrace{\nu = \frac{\hbar}{2M}}_{\text{quantum fluctuation}} \qquad \varepsilon(\rho_M) = \underbrace{V(\vec{x}) \frac{\rho_M}{M}}_{\text{external force}} + \underbrace{\frac{1}{2} U_0 \left(\frac{\rho_M}{M} \right)^2}_{\text{two-body interaction}}$$

Application of SVM

Koide&Kodama, JPA45, 255204 (`12)

$$i\hbar\partial_t\varphi = \left[-\frac{\hbar^2}{2M}\Delta + V + U_0 |\varphi|^2 \right]\varphi$$

Gross-Pitaevskii equation



BEC

Nucleation

Dissipative Dynamics

Stochastic Representation of Action

Classical action
$$I_{cla} = \int_a^b dt \left(\frac{m}{2} \left(\frac{d\vec{r}(t)}{dt} \right)^2 - V(\vec{r}(t)) \right)$$

For example

$$\left(\frac{d\vec{r}}{dt} \right)^2 \Rightarrow \begin{cases} 1) \quad D\vec{r} \cdot D\vec{r} \\ 2) \quad \tilde{D}\vec{r} \cdot \tilde{D}\vec{r} \\ 3) \quad \dfrac{D\vec{r} \cdot D\vec{r} + \tilde{D}\vec{r} \cdot \tilde{D}\vec{r}}{2} \end{cases}$$

We consider 3)

Stochastic action
$$I_{sto} = \int_a^b dt E\left[\frac{m}{2} \frac{(D\vec{r})^2 + (\tilde{D}\vec{r})^2}{2} - V(\vec{r}) \right]$$

Koide, JP Conf. 410, 012025 (`13)

Replacement of kinetic terms

The general expression of $\left(d\vec{r}(\vec{R},t)/dt\right)^2$ is given by

$$\boxed{A\left(D\vec{r}\right)^2 + B\left(\tilde{D}\vec{r}\right)^2 + C\left(D\vec{r}\right)\left(\tilde{D}\vec{r}\right)}$$

For noise=0, the classical kinetic term should be reproduced.

$$\left(d\vec{r}(\vec{R},t)/dt\right)^2 \Rightarrow$$
$$\left(\frac{1}{2}+\alpha_2\right)\left\{\left(\frac{1}{2}+\alpha_1\right)\left(D\vec{r}\right)^2 + \left(\frac{1}{2}-\alpha_1\right)\left(\tilde{D}\vec{r}\right)^2\right\} + \left(\frac{1}{2}-\alpha_2\right)\left(D\vec{r}\right)\left(\tilde{D}\vec{r}\right)$$

The calculation so far corresponds to $(\alpha_1,\alpha_2)=(0,1/2)$.

Koide, JP Conf. 410, 012025 (`13)

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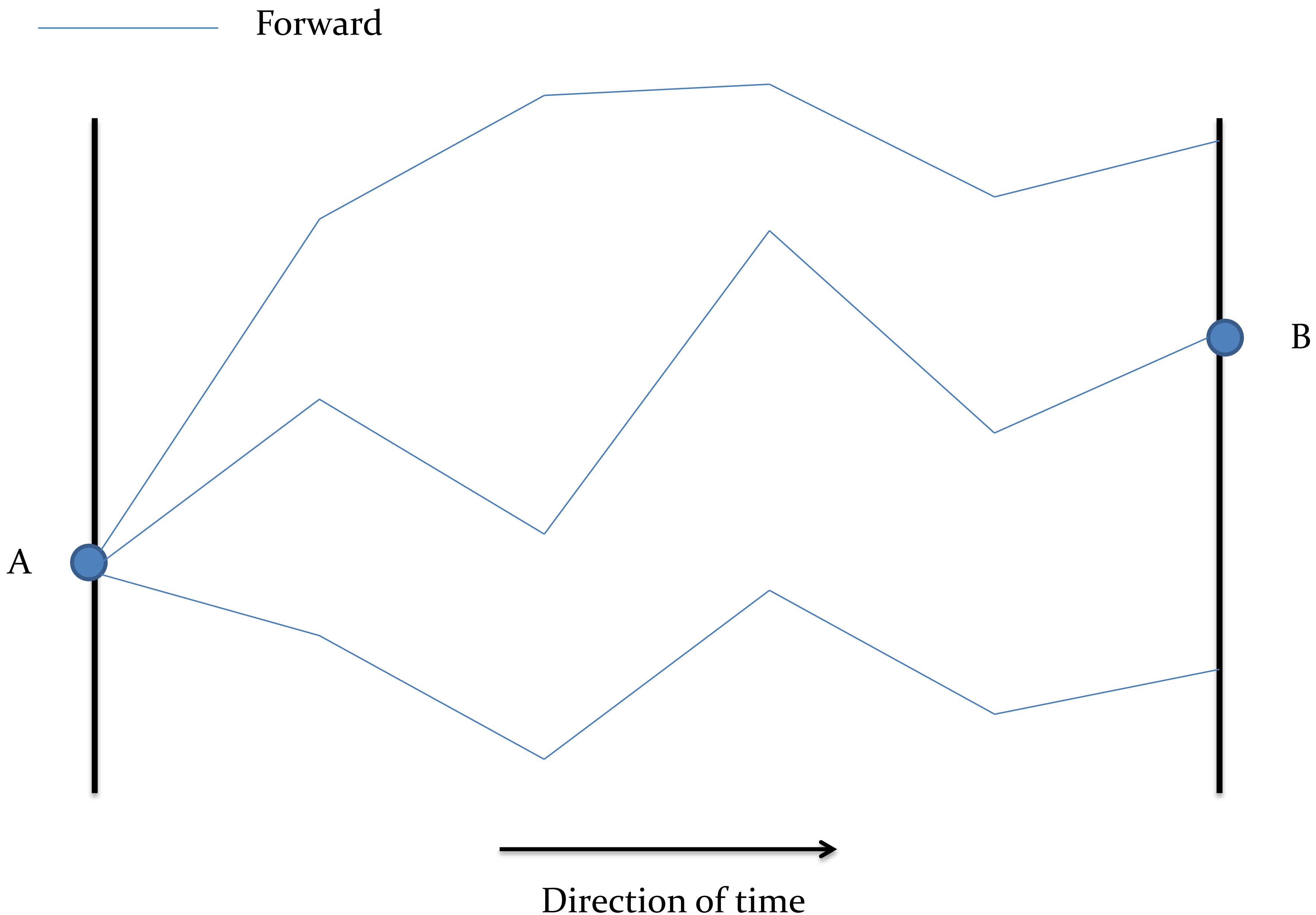
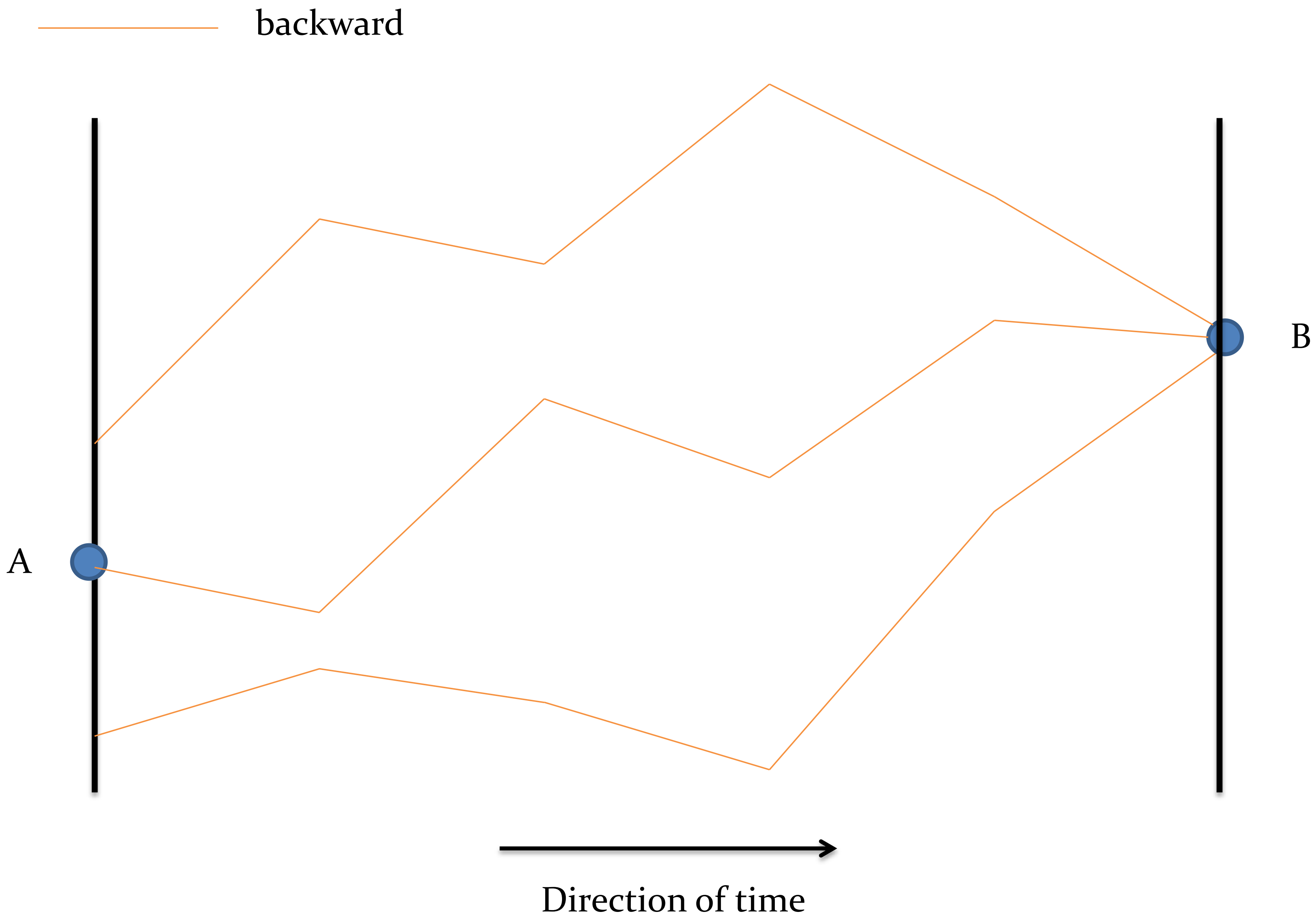
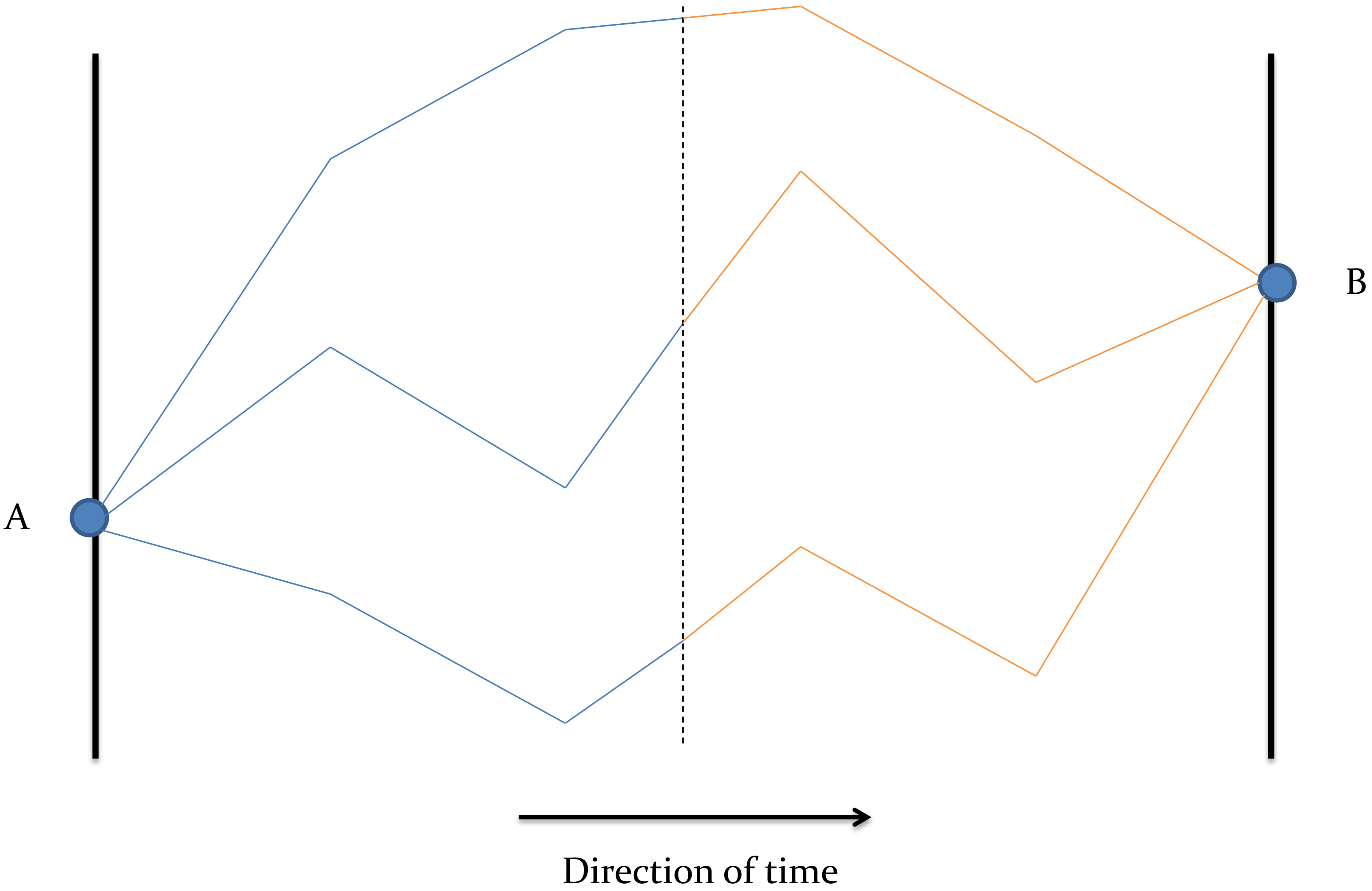
The calculation so far corresponds to $(\alpha_1,\alpha_2)=(0,1/2)$.

If α_1 is not zero, the time-reversal sym. is violated (NSF eq.).

Another Explanation


Forward
A
B
Direction of time

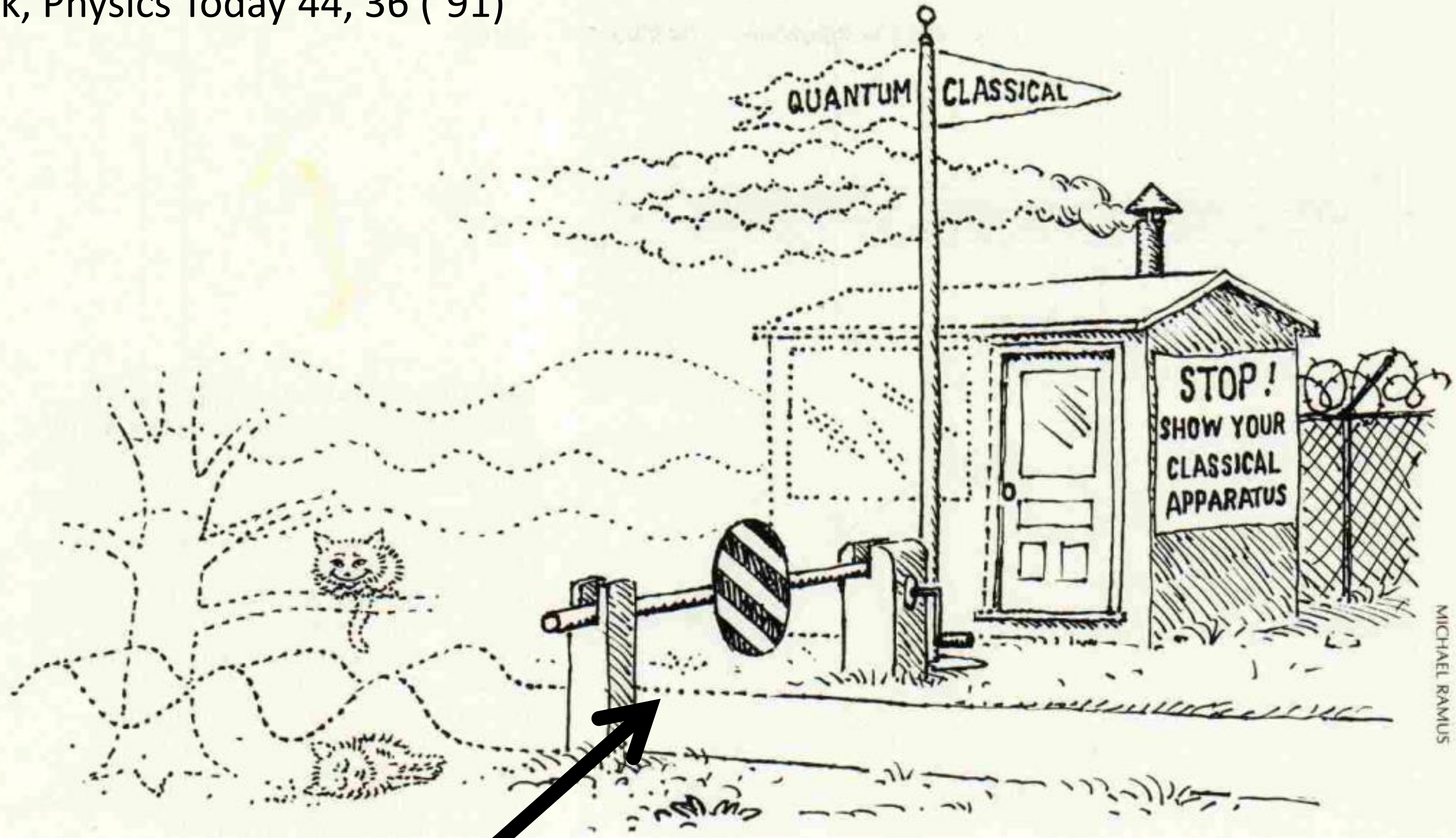

backward
A
B
Direction of time



See also the Bernstein process.

Quantum-Classical hybrids

Zurek, Physics Today 44, 36 (`91)



Delineating the border between the quantum realm ruled by the Schrödinger equation and the classical realm ruled by Newton's laws is one of the unresolved problems of physics. **Figure 1**

Near the boundary, we can expect a coexistence of Class. and Quan. degrees of freedom.

- Quantum measurement
- Quantum-to-classical transition in early universe
- Einstein gravity interacting with quantum objects
- Simplification of complex simulation of quantum chemistry

Models of quantum-classical hybrids should satisfy,

- Energy conservation
- Positivity of probability
- Newton equation + Schrödinger equation in no int. V=0.

Most of proposed models cannot satisfy these conditions.

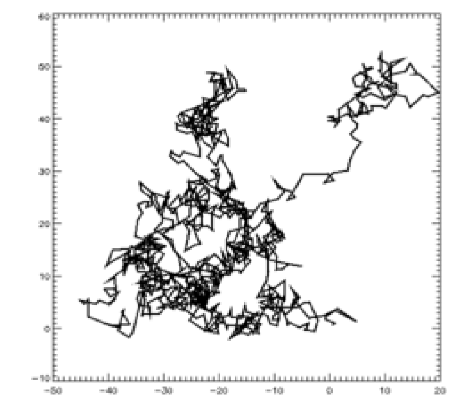
Quantum Two Particles Lagrangian

$$L = E\left[\frac{M}{2}\frac{(D\vec{r}_q)^2 + (\tilde{D}\vec{r}_q)^2}{2} + \frac{m}{2}\frac{(D\vec{r}_q)^2 + (\tilde{D}\vec{r}_q)^2}{2} - V(\vec{r}_q, \vec{r}_q)\right]$$

$$d\vec{r}_q = \vec{v}dt + \sqrt{\frac{\hbar}{M}}d\vec{W}$$

$$d\vec{r}_q = \vec{u}dt + \sqrt{\frac{\hbar}{m}}d\vec{W}'$$

Stochastic variables



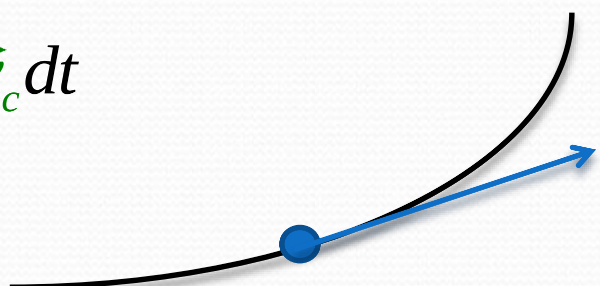
Hybrid Lagrangian

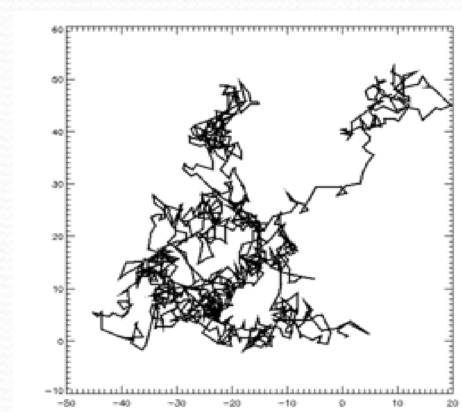
$$L = E\left[\frac{M_c}{2}\frac{(D\vec{r}_c)^2 + (\tilde{D}\vec{r}_c)^2}{2} + \frac{m_q}{2}\frac{(D\vec{r}_q)^2 + (\tilde{D}\vec{r}_q)^2}{2} - V(\vec{r}_c, \vec{r}_q)\right]$$

Classical variable

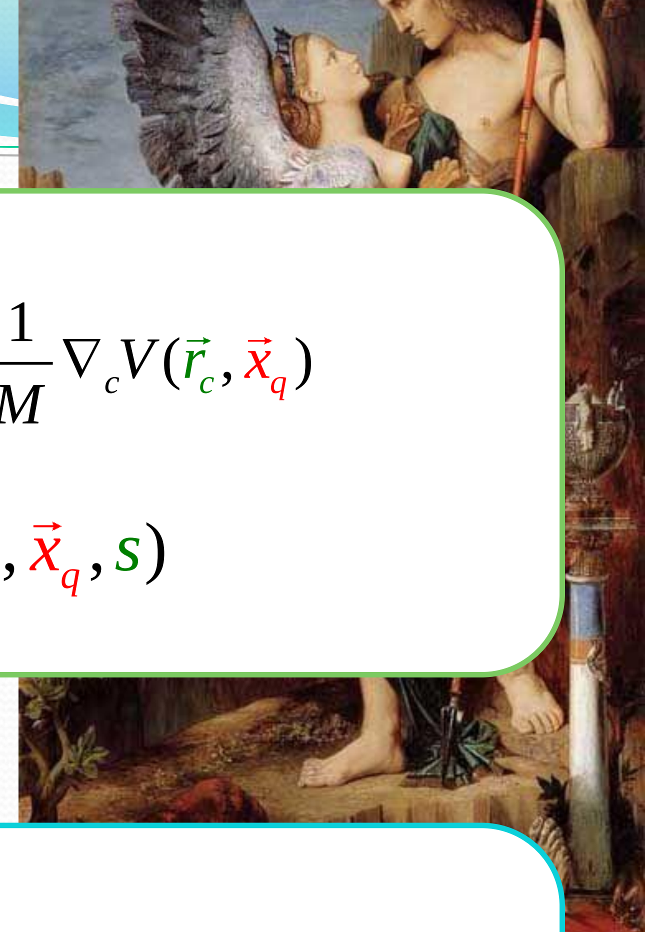
Stochastic variable

$$d\vec{r}_c = \vec{v}_c dt$$





$$d\vec{r}_q = \vec{u}dt + \sqrt{\frac{\hbar}{m}}d\vec{W}$$



Classical Variation

$$\left(\frac{d}{dt}+\underline{\frac{\hbar}{m}\nabla_q \vartheta(\vec{r}_c,\vec{x}_q,t)\cdot\nabla_q}\right)\vec{v}_c(\vec{r}_c,\vec{x}_q,t)=-\frac{1}{M}\nabla_c V(\vec{r}_c,\vec{x}_q)$$

$$\vec{r}_c(\vec{x}_q,t)=\vec{x}_{c0}+\int_{t_0}^{t} ds\ \vec{v}_c(\vec{r}_c(\vec{x}_q,s),\vec{x}_q,s)$$

Stochastic Variation

$$i\hbar\underline{\frac{d}{dt}}\varphi_q(\vec{r}_c,\vec{x}_q,t)=\left[-\frac{\hbar^2}{2m}\Delta_q+V(\vec{r}_c,\vec{x}_q)\underline{-\frac{M}{2}\vec{v}_c^2(\vec{r}_c,\vec{x}_q,t)}\right]\varphi_q(\vec{r}_c,\vec{x}_q,t)$$

$$\varphi_q(\vec{r}_c,\vec{x}_q,t)=\sqrt{\rho(\vec{x}_q,t)}e^{i\vartheta(\vec{r}_c,\vec{x}_q,t)}$$

Koide, PLA379 2007 (2015)

It was confirmed

Check list by Caro, Diósi, Elze et al.

O.K. ◆ Energy conservation ⟵ Stochastic Noether theorem

O.K. ◆ Positivity of probability

O.K. ◆ Newton equation + Schrödinger equation in no int. V=0.

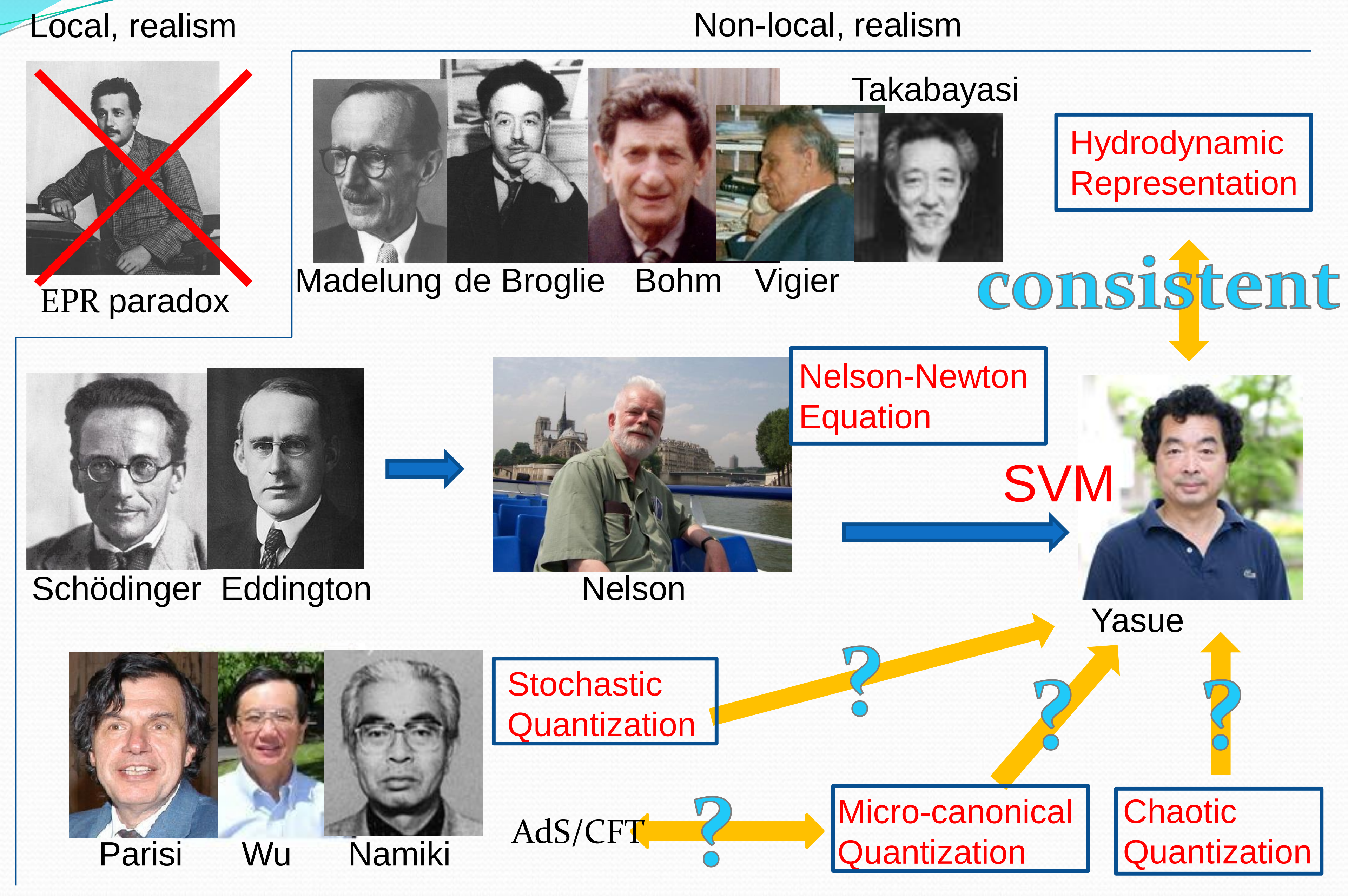
O.K. ◆ Generalized Ehrenfest theorem

Other successful models,

- Hall&Reginatto, PRA72, 062109 (`05)
- Elze, PRA85, 052109 (`12);Lampo et al., PRA90, 042120 (`14) , Radonji et al., PRA85, 064101 (`12)

Non conventional formulations

Non Conventional Formulation of QM



Nelson's Stochastic Quantization

SVM is the reformulation of Nelson's approach in the framework of a variational principle.

In the original Nelson's approach, quantization indicates

$$m\frac{d^2x}{dt^2} = -\partial_x V(x) \quad \xrightarrow{\text{REPLACE}} \quad m\frac{D\tilde{D}x + \tilde{D}Dx}{2} = -\partial_x V(x)$$

Nelson-Newton equation

Nelson's Stochastic Quantization

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REPLACE

Nelson-Newton equation

$$m\frac{D\tilde{D}x + \tilde{D}Dx}{2} = -\partial_x V(x)$$

In SVM, this is obtained by the optimization of action.

Parisi-Wu's Stochastic Quan.(I)

Damgaard&Hueffel, PR. 152, 227 (`87)
Krein et al., IJMP A29, 1450030 (`14)

Wick rotation	$S[\phi]$	$\longrightarrow$	$S_E[\phi]$
	4 D		4+1 D
Intro. of a virtual time	$\phi(x_E^\mu)$	$\longrightarrow$	$\phi(x_E^\mu, \tau)$

We consider the stochastic motion in this virtual time,

SDE $$d\phi(x_E^\mu,\tau) = -\frac{\delta S_E[\phi]}{\delta\phi(x_E^\mu,\tau)}d\tau + \sqrt{2}dW(x_E^\mu,\tau)$$

Wiener Process

Parisi-Wu's Stochastic Qua.(II)

For the free case, $S_E[\phi]=\frac{1}{2}\int d^4k_E\tilde{\phi}(k_E^\mu,\tau)\left[k_E^2+m^2\right]\tilde{\phi}(k_E^\mu,\tau)$

we can solve the SDE as

$$\tilde{\phi}(k_E^\mu,\tau)=e^{-\left[k_E^2+m^2\right]\tau}\tilde{\phi}(k_E^\mu,0)+\int_0^\tau ds\sqrt{2}e^{-\left[k_E^2+m^2\right](\tau-s)}\frac{dW(k_E^\mu,s)}{ds}$$

Propagator

$$G(k_E^\mu,p_E^\mu)\equiv\lim_{\tau=\tau'\to\infty}E\left[\tilde{\phi}(k_E^\mu,\tau)\tilde{\phi}(p_E^\mu,\tau')\right]$$

$$=(2\pi)^4\delta^{(4)}(k_E^\mu-p_E^\mu)\frac{1}{k_E^2+m^2}$$

These successes are just accidental?

FERMION is a biggest open question!

As a pedagogical review, Koide, Kodama&Tsushima, JP Conf. 626, 012055 (`15)
http://iopscience.iop.org/1742-6596/626/1