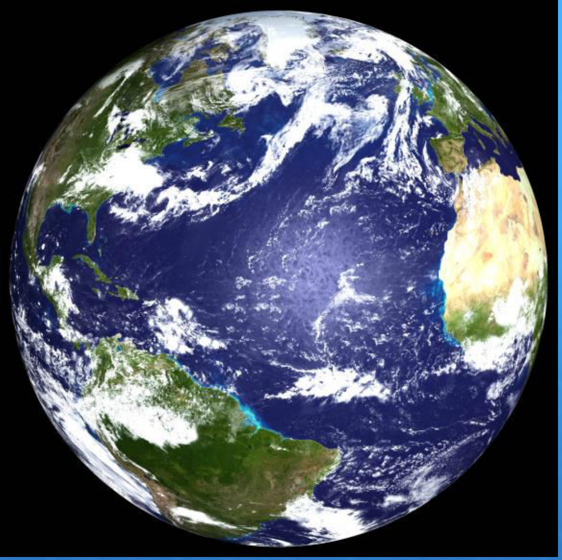


Invariant Mathematical Structures & Angular Momentum in Ocean Dynamics and Implied drift of Oceanic Monopolar Planetary Vortices

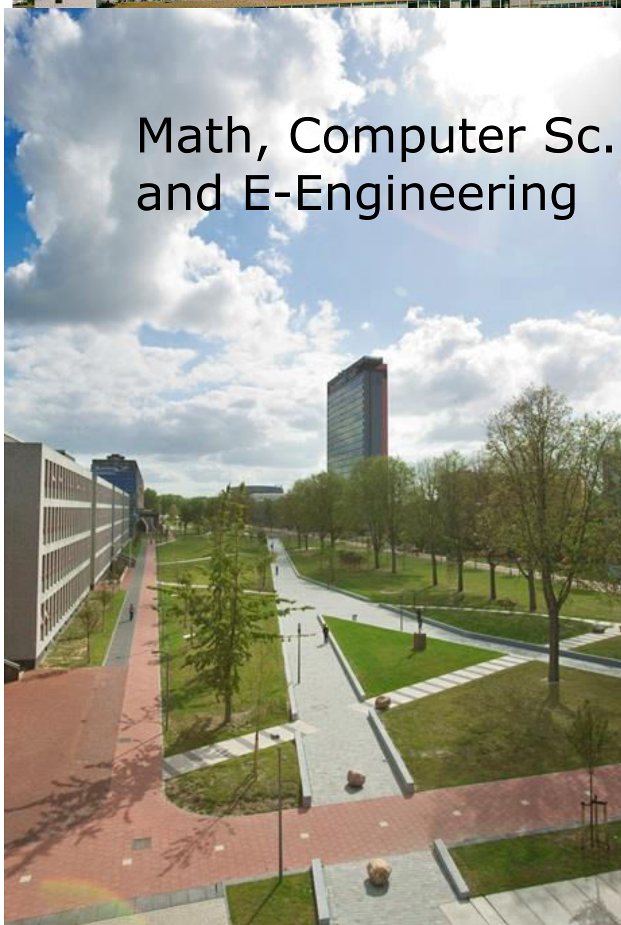
Ramses van der Toorn
Delft University of Technology

The Netherlands: water land

Netherlands Institute for Sea Research

Oosterscheldekering (1986)

Delft University of Technology



Math, Computer Sc.
and E-Engineering

Maritime Engineering

Geo- and Civil Engineering

Overall Research Theme

Math, Computer Sc. and E-Engineering

Application / Applicability of Math
to
Conceptual Understanding
of
Processes in the Oceans

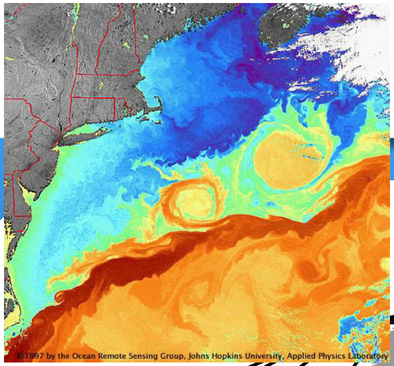
This Presentation

Math, Computer Sc. and E-Engineering

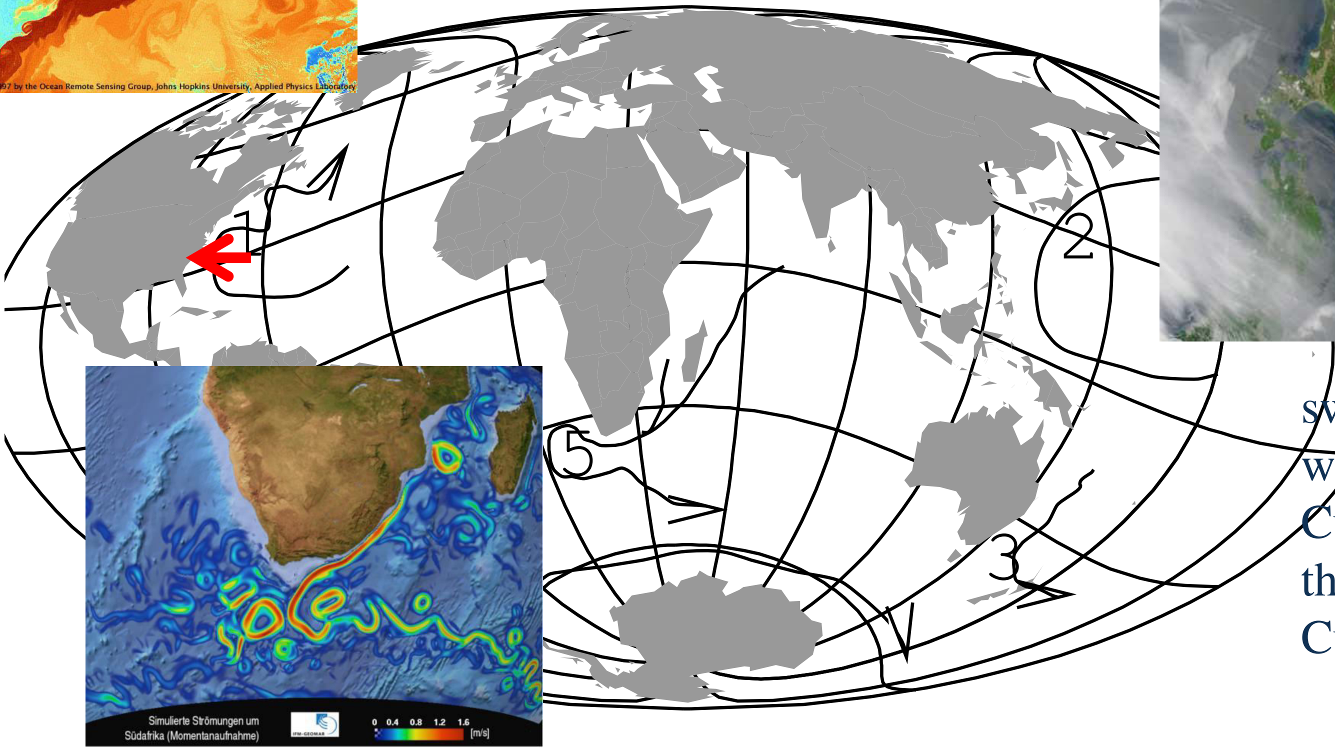
Application / Applicability of Math
to
Conceptual Understanding
of
Processes in the Oceans

Intrinsic Drift of Monopolar Vortices

Mesoscale Oceanic Monopoles



Gulf Stream Rings...



swirling phytoplankton where warm Kuroshio Current collides with the frigid Oyashio Current

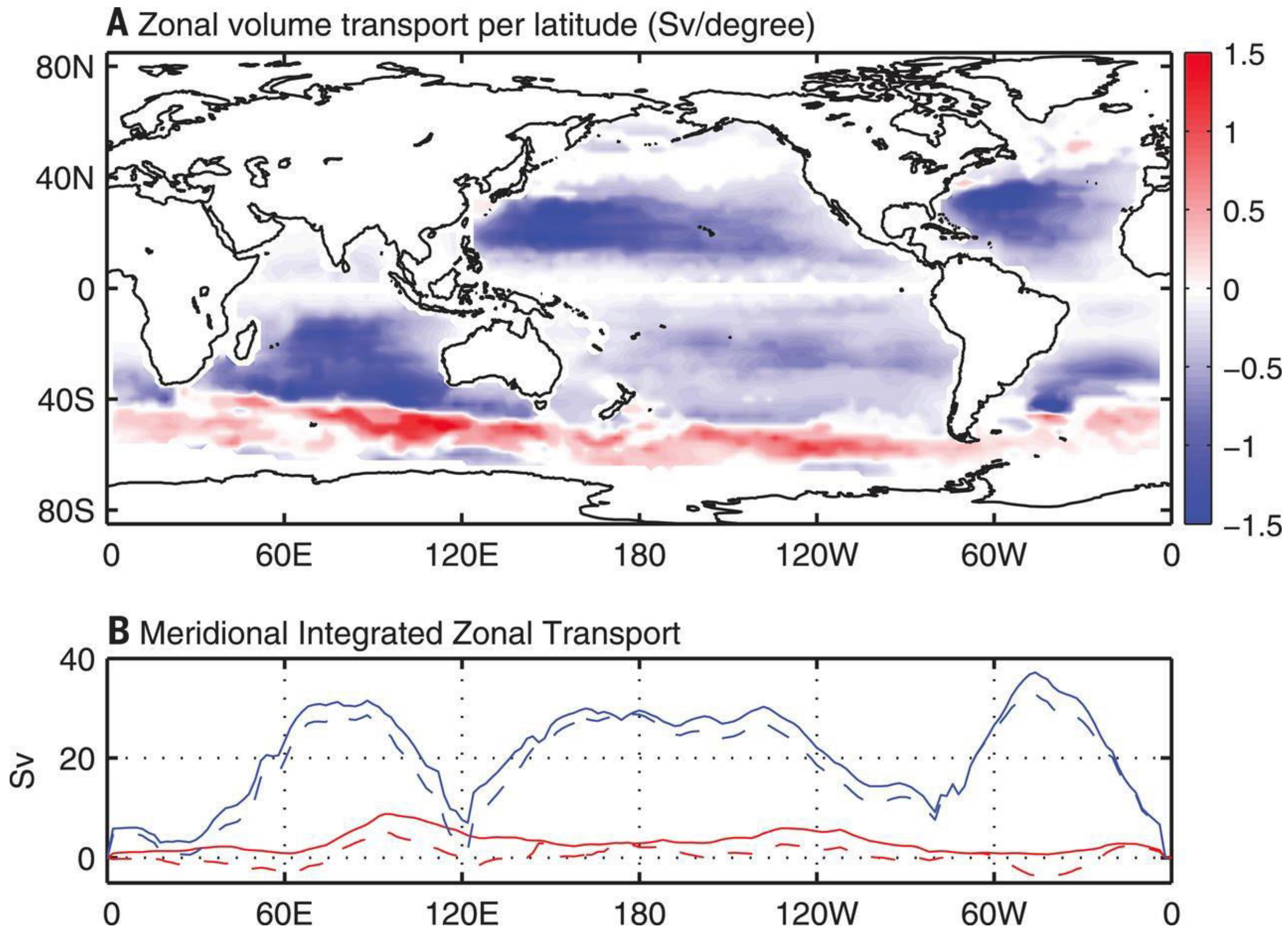
Ring shed from Agulhas current

Strong Western Boundary Currents Shed Strong Oceanic Vortices

Mesoscale Oceanic Monopoles

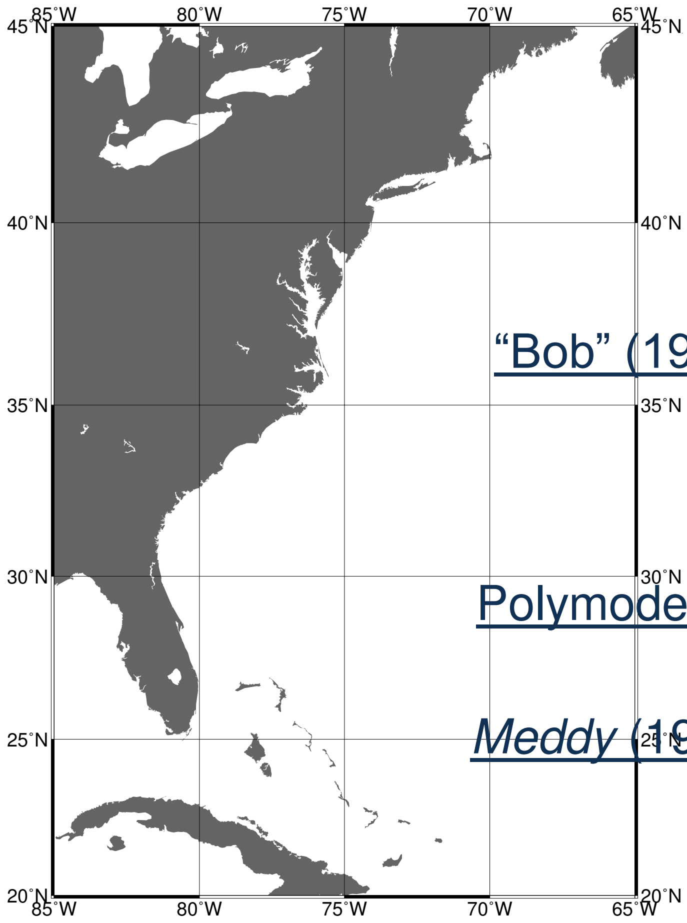
Observed zonal transport by mesoscale eddies in the oceans:

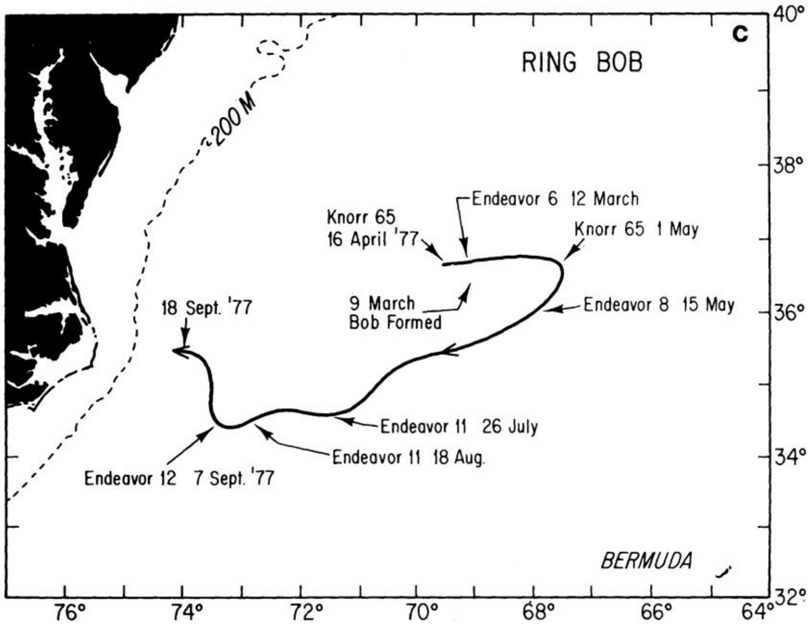
"..magnitude comparable to large scale wind- and thermohaline-driven circulations."



(Zhang & Qiu, Science, June 2014)

Intrinsic Drift; Observations



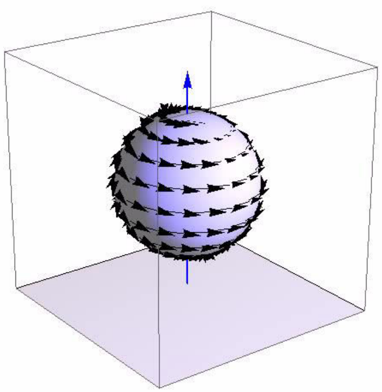


The Soviet Polymode experiment ('77 – '78) directly demonstrated ***intrinsic drift of oceanic vortices***: **1 .. 6 cm/s**.

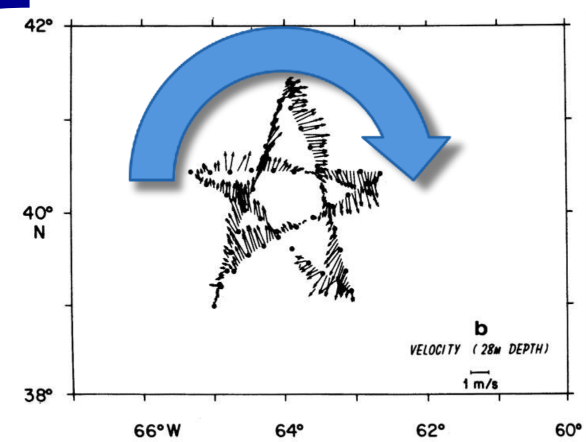
Scientific Method

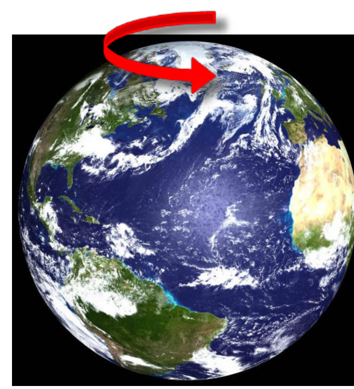
Applied **Math /**
Mathematical Physics

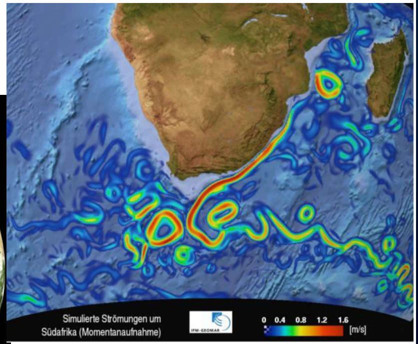
Mathematics
a
Science of ***Structures***,
as such



?





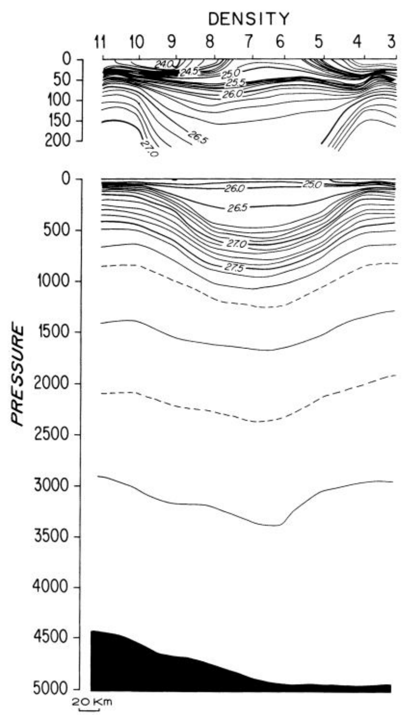


Sc. Method: "Mathematical Structures 1"

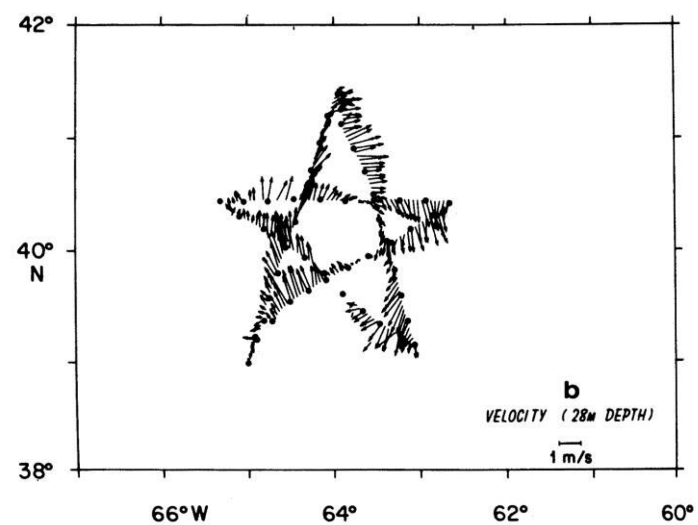
Simple Mathematical Structures in the Ocean Sciences:

Directly observable:

- scalar fields:
  density distributions
- vector fields:
  velocity distributions



Structure of an Anti-Cyclonic Gulf Stream Ring (Joyce, JPO, 1984)



Sc. Method: "Mathematical Structures 2"

Fundamental **Mathematical Structures in the Ocean Sciences:**

"Laws of Nature":

Universal, preserved, obeyed patterns in the apparently ever changing world

⇩

Differential Equations :

- conservation laws:
    - e.g. mass
- dynamics:
    - e.g: momentum balance

Sc. Method: "*Implications of the Laws*"

Fundamental Mathematical Structures in the Ocean Sciences:

"Laws of Nature":

Universal, preserved, obeyed patterns in the apparently ever changing world

Differential Equations :

*: "Logik der Forschung", Karl Popper (1935)

- ***Mathematics of solving DE's:***
  - study the implications of the Laws
  - show observed ***functions*** obeys them indeed (i.e. *test** of our **knowledge** of the physical world)

Mathematical Structures 3

Deeper **Mathematical Structures in the Ocean Sciences:**

"Invariants":

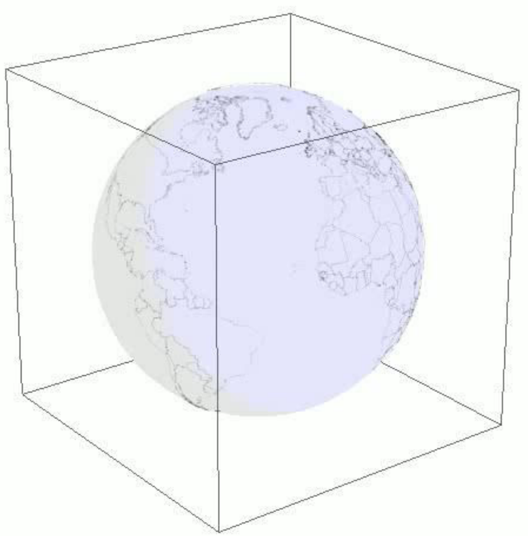
- Remark: ***functions*** describing observable fields ***change*** with change of coordinate system
- behind these different ***descriptions***, are the ***invariable, independently existing entities***.

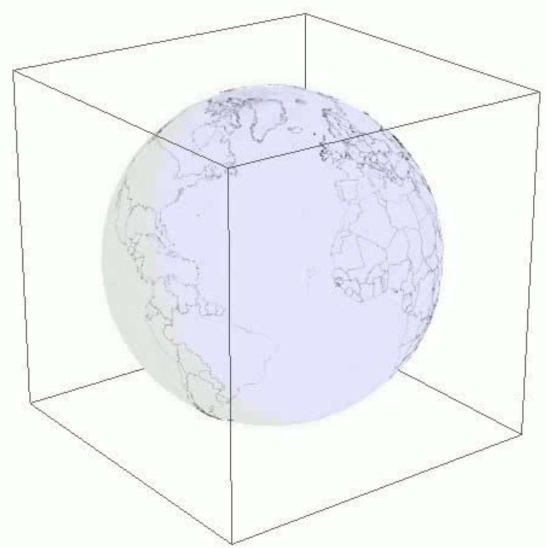
Geometrical Objects :

- ***branches of Mathematics:***
  - general tensor calculus
  - group theory
  - ..

Mathematical Structures 3

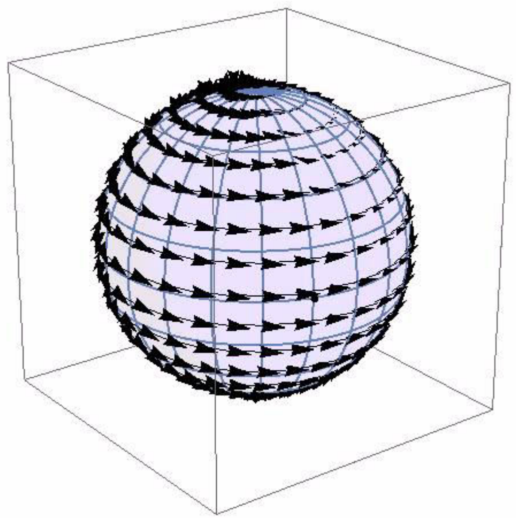
Deeper Mathematical Structures in the Ocean Sciences:

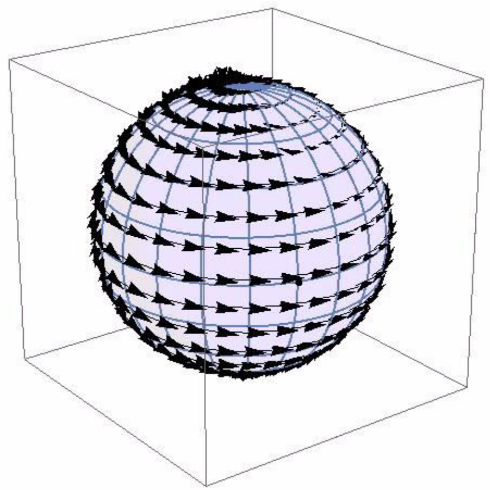




Game: change coordinates and construct objects that don't change along.

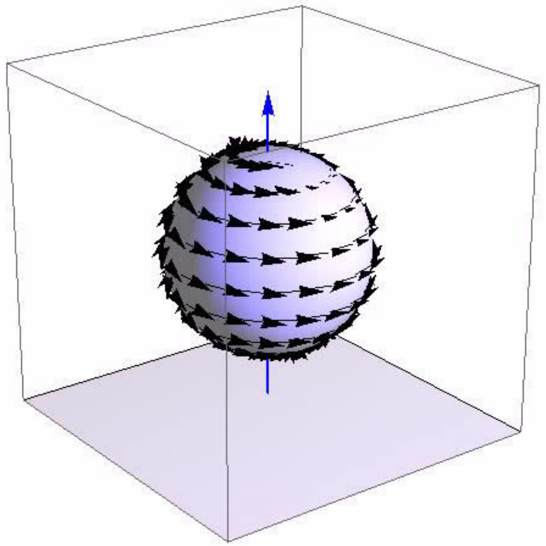
Invariants





Invariants: what lies *behind* the coordinate representations??

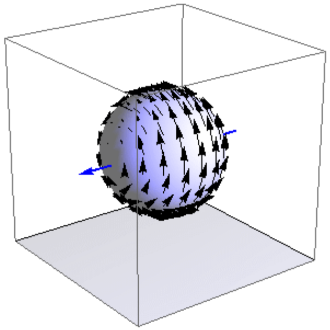
Symmetry of a Sphere

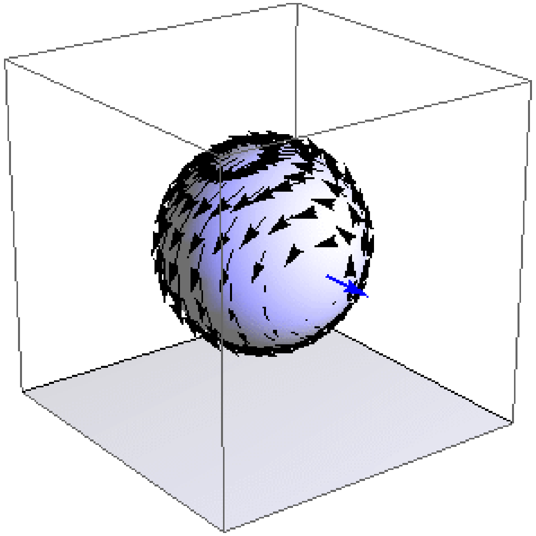


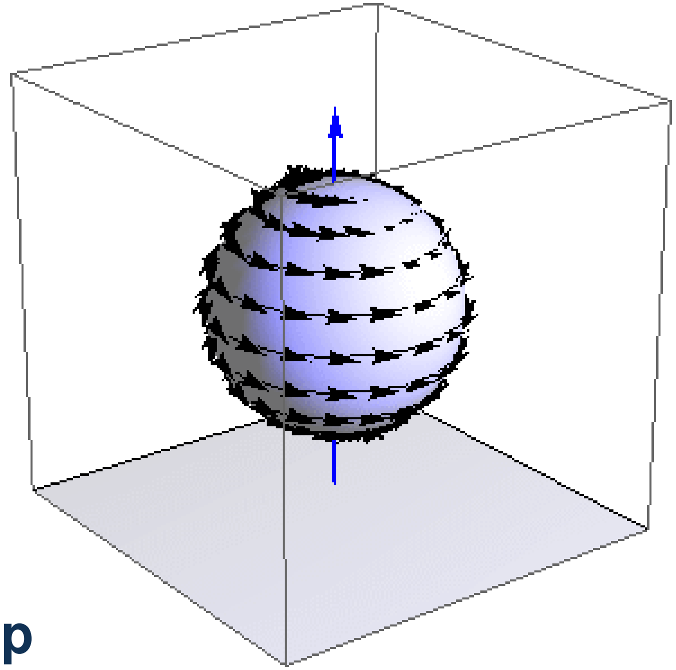
A Sphere is symmetrical

- Shape is *invariant*, when we rotate it.
- An infinite sea of possible rotations exists.
    - We can e.g. choose from an infinite amount of different orientations of the axis of rotation.

Lie Group of Rotations



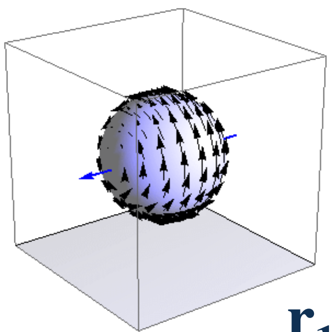




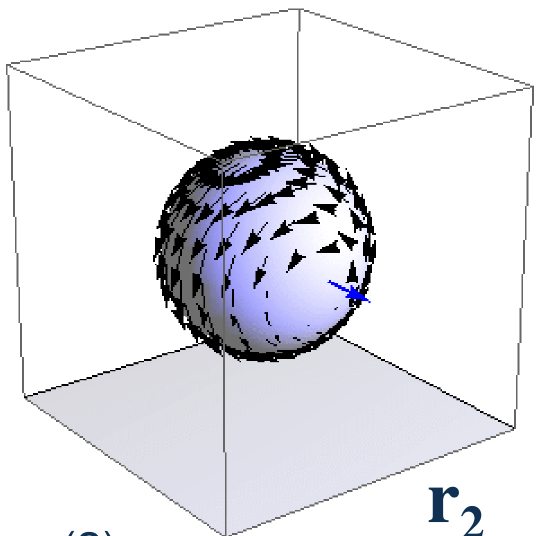
Rotations form a **Lie group**

Sophus Lie (1842 – 1899)

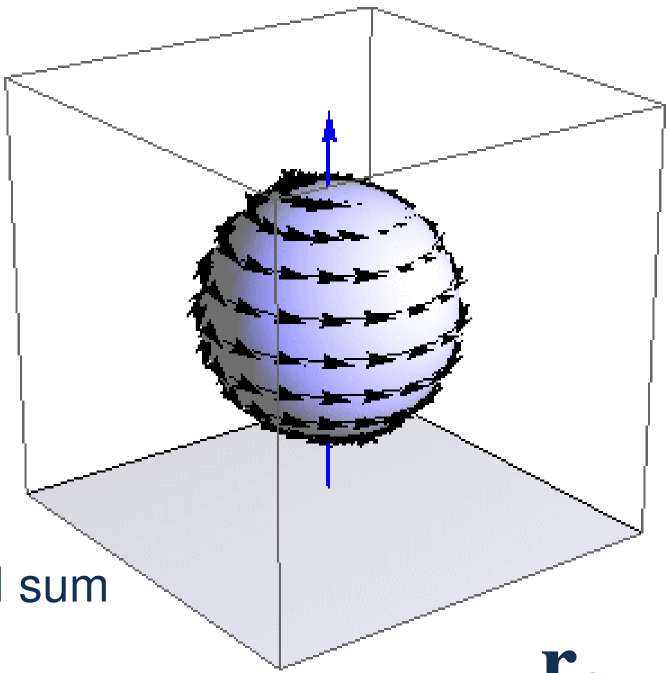
Lie Group of Rotations



$\mathbf{r}_1$



$\mathbf{r}_2$



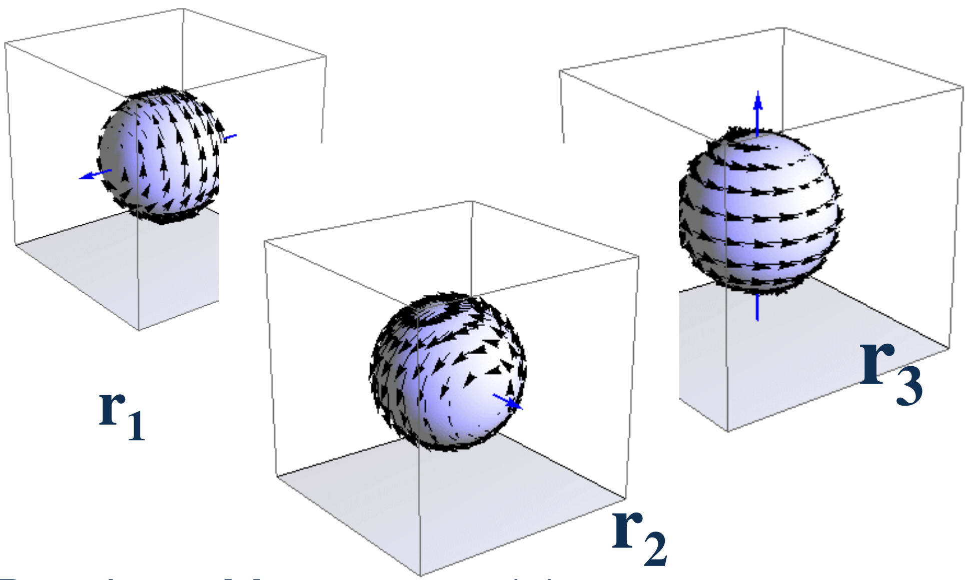
$\mathbf{r}_3$

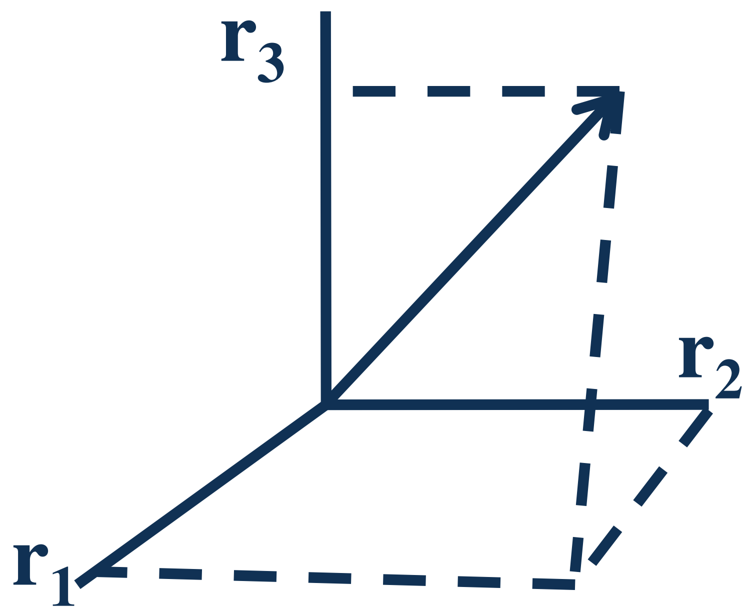
Rotations: **Lie group** so(3):

- *All* rotations can be generated from a weighted sum of 3 basic generators
- product **[.,.]: $[\mathbf{r}_1, \mathbf{r}_2] = -\mathbf{r}_3$** ….. and cyclic

  => "Lie Algebra"
- **=> closed structure, under this sum and product,**

Lie Group of Rotations

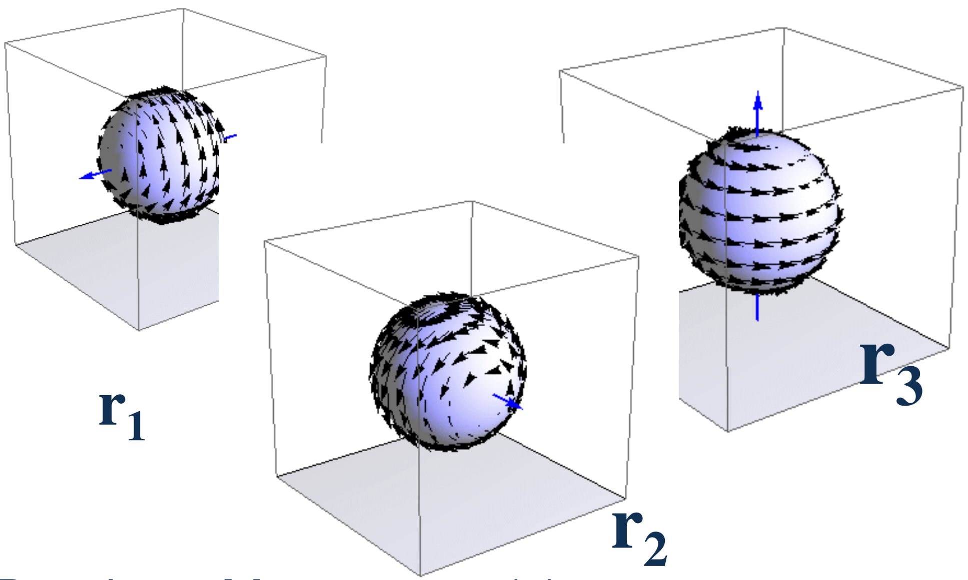


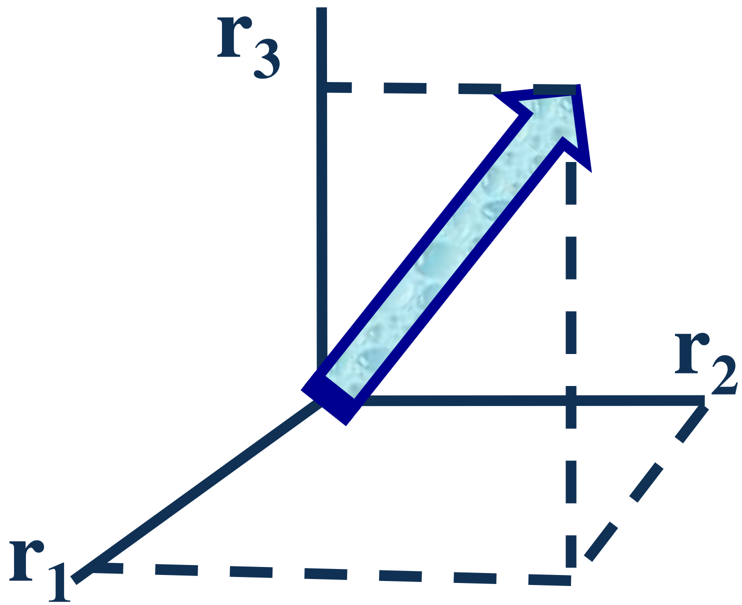


Rotations: **Lie group** so(3):

- *All* rotations can be generated from a weighted sum of 3 basic generators (=> <u>*3-dimensional*</u>)
- Lie product **[.,.]: $[r_1, r_2] = -r_3$** ….. and cyclic **=> "Lie Algebra"**
- inner product: **$\langle r_i, r_j \rangle = \delta_{ij}$** **=> enables projection of fields onto the so(3) algebra ...**

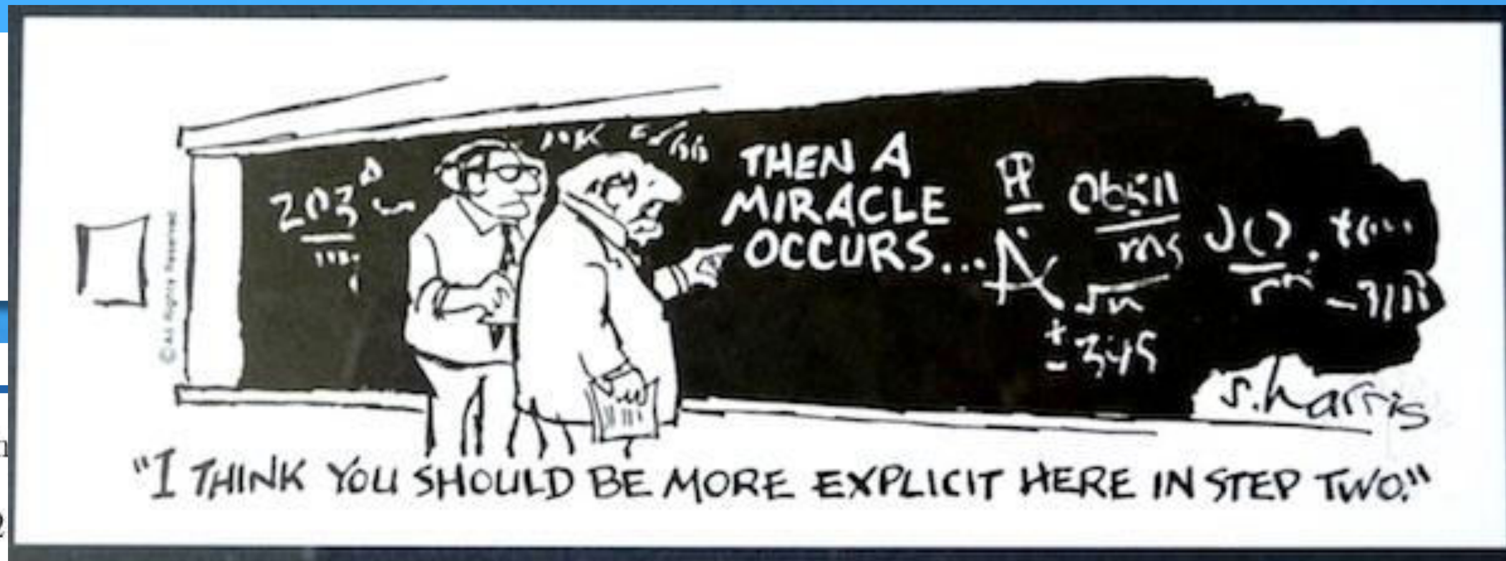
Lie Group of Rotations





Rotations: **Lie group** so(3):

- inner product: **$\langle r_i, r_j \rangle = \delta_{ij}$** **=> enables *projection* of fields onto the so(3) algebra ...**
- **so *any* vector field has 3 independent so(3) components**
- **one can construct the dynamics of these!**



$$\ddot{\lambda}\ \left(\cos(\theta)^2 m + \sin \ldots\right.$$
$$\dot{\theta}\ \Big(-\sin(2\theta)(\Omega \ldots$$
$$+(\Omega+\dot{\lambda})\sin(\theta)^2\frac{dm_{r^2}}{dt}+\sin(\theta)\frac{d}{dt}\{\tau_0 \parallel \rho\,\breve{v}\} \;=0. \quad (42)$$

$$\ddot{\lambda}\ \left(\frac{1}{2}\sin(2\theta)(m_{r^2}-m)\right)+$$
$$\dot{\theta}\,[(\Omega+\dot{\lambda})\,(2\ \sin(\theta)^2\,m+\cos(\theta)^2 m_{r^2})$$
$$-\sin(\theta)\,\{\tau_0 \parallel \rho\,\breve{v}\}] \quad (43)$$
$$+\frac{1}{2}(\Omega+\dot{\lambda})\sin(2\theta)\frac{dm_{r^2}}{dt}+\cos(\theta)\frac{d}{dt}\{\tau_0 \parallel \rho\,\breve{v}\} \;=0.$$

$$m\ddot{\theta}+\frac{1}{2}m\sin(2\theta)\dot{\lambda}\,(\dot{\lambda}+2\Omega) \;= \quad (44)$$
$$(\Omega+\dot{\lambda})\ \cos(\theta)\,\{\tau_0 \parallel \rho\,\breve{v}\}.$$

- Mathematical formulation:
  - **3** equations (ODE): one for each basis element of the so(3) Lie algebra:
  - momentum balance is *projected* on these
  - Physics: Integral Angular Momentum Equations.

(v.d Toorn & Zimmerman; *Journal of Mathematica Physics*, August 2010)

The Angular Momentum Balance

$$\ddot{\lambda}\ \left(\cos(\theta)^2 m + \sin(\theta)^2 m_{r^2}\right) +$$
$$\dot{\theta}\left(-\sin(2\theta)(\Omega+\dot{\lambda})\left(m - \frac{1}{2}m_{r^2}\right) + \cos(\theta)\left\{\tau_0 \parallel \rho\,\breve{v}\right\}\right)$$
$$+(\Omega+\dot{\lambda})\sin(\theta)^2\frac{dm_{r^2}}{dt} + \sin(\theta)\frac{d}{dt}\left\{\tau_0 \parallel \rho\,\breve{v}\right\} = 0. \quad (42)$$

$$\ddot{\lambda}\ \left(\frac{1}{2}\sin(2\theta)(m_{r^2} - m)\right) +$$
$$\dot{\theta}\left[(\Omega+\dot{\lambda})\left(2\ \sin(\theta)^2\, m + \cos(\theta)^2 m_{r^2}\right)\right.$$
$$\left. -\sin(\theta)\left\{\tau_0 \parallel \rho\,\breve{v}\right\}\right] \quad (43)$$
$$+\frac{1}{2}(\Omega+\dot{\lambda})\sin(2\theta)\frac{dm_{r^2}}{dt} + \cos(\theta)\frac{d}{dt}\left\{\tau_0 \parallel \rho\,\breve{v}\right\} = 0.$$

$$m\ddot{\theta} + \frac{1}{2}m\sin(2\theta)\dot{\lambda}\,(\dot{\lambda}+2\Omega) = \quad (44)$$
$$(\Omega+\dot{\lambda})\ \cos(\theta)\left\{\tau_0 \parallel \rho\,\breve{v}\right\}.$$

- Mathematical formulation:
  - **3** equations (ODE): one for each basis element of the so(3) Lie algebra:
  - momentum balance is *projected* on these
  - Physics: Integral Angular Momentum Equations.

(v.d Toorn & Zimmerman; *Journal of Mathematica Physics*, August 2010)

Results: Velocities, Trajectories.

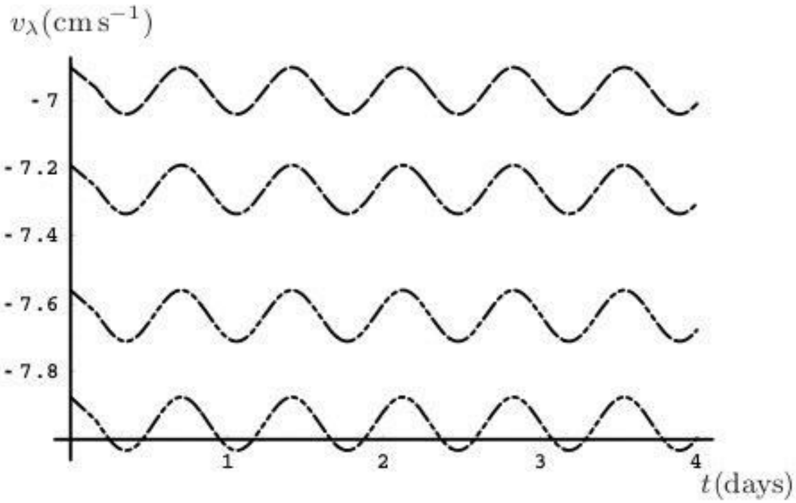


FIG. 2: Longitudinal velocities v_λ $(\mathrm{cm\,s^{-1}})$ as a function of time t (days) of, from top to bottom, a cyclo-geostrophic cyclone, a geostrophic cyclone, a geostrophic anticyclone and a cyclo-geostrophic anticyclone. All these vortices had a Gaussian profile and obey the full angular momentum equations

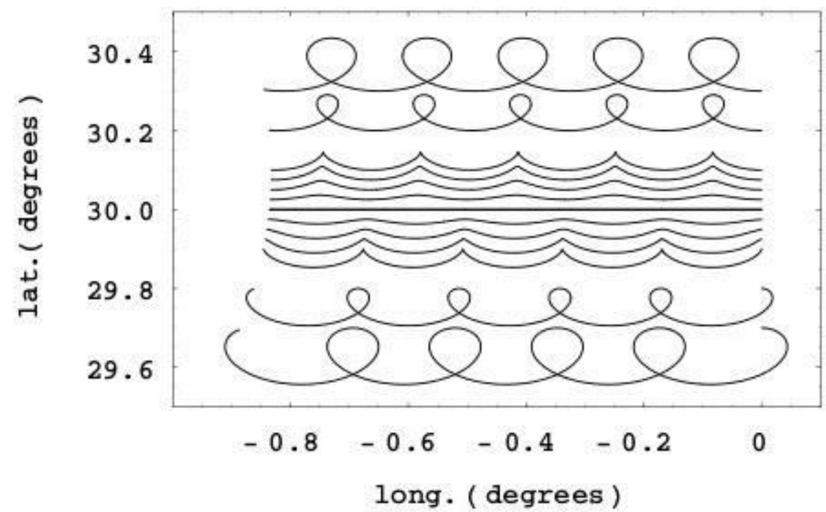
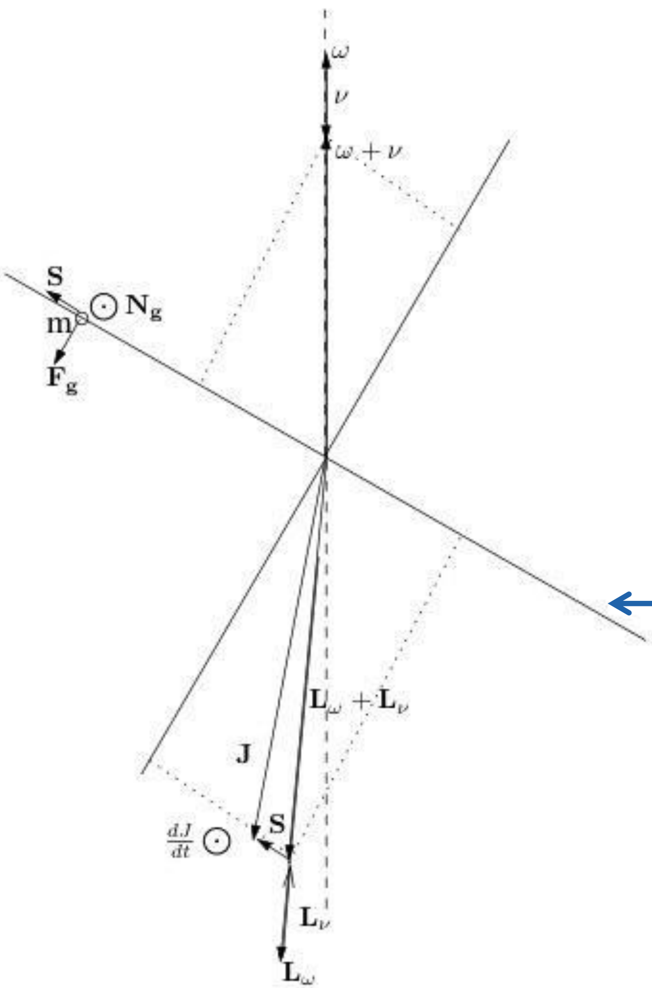
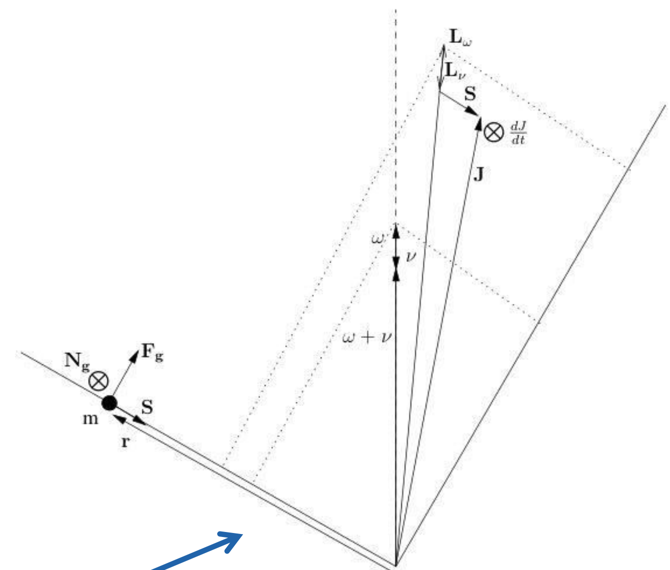


FIG. 3: Trajectories of several geostrophic anti-cylones as found by numerical integration. At $t = 0$ the vortices are released at several latitudes near $\theta = 30^o$. They also have different initial longitudinal velocities. The vortex that starts at 30^o has initial velocity λ_0' as given by equation (63). Overall the vortices have initial velocities $(1 + c)\lambda_0'$, where, from bottom to top, $c = -3, -2, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 2, 3$.

(v.d Toorn & Zimmerman; *Journal of Mathematica Physics*, August 2010)

Ocean Vortex Angular Momentum Balance





Cyclones vs. Anti-Cyclones

- Opposite Orientation of Spin
- Opposite sign of (surplus) Mass
  - Low Pressure vs. High pressure cores!

⇒ All go Westwards

⇒ (as opposed to the gyroscope)

Inventory of Results

1. Reproduced Intrinsic Westward Drift of Oceanic Monopoles
   a. right direction, correct speed,
      including well known integrals in GFD lit. for this speed
   b. global result: "westward" = along latitude circles

2. <u>Demonstrated</u> that 1 is implied by Classical Fluid Mechanics,
   projected onto so(3).
3. Results at a deeper level:

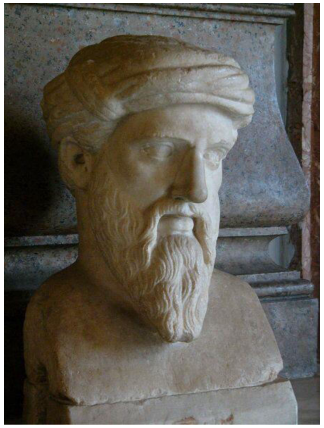
   1. identified relevant *invariants* (metric, isometry group)
   2. shown how these form building blocks for the "laws of nature"
   3. show how all this *implies* vortex drift
   4. unified *vortex drift* with *rigid body mechanics* ("spinning tops")


Thank you. Any questions?

"The Problem of Scientific Knowledge*"

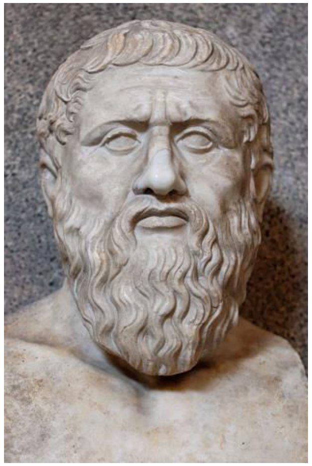
Mathematical Structures in the Ocean Sciences:

Constant Quantities



and Qualities.. **..in the real world, which is *in flux*.**

**: Pythagoras (Πυθαγόρας) (572 – 500 BC)

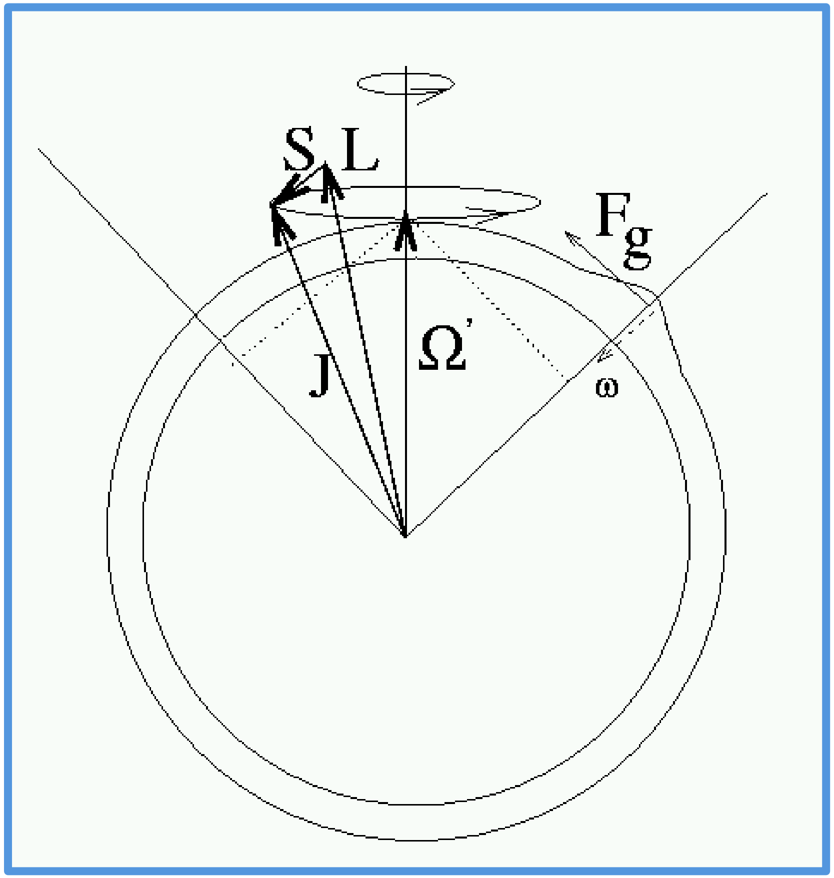


Plato (427 - 347 BC.)

Heraclitus (Ephesus,535 – c. 475 BC), by Johannes Moreelse(1603-1634)

*cf, eg: "Logik der Forschung", Karl Popper (1935)

The Full Picture: Vortex on a Rotating Sphere



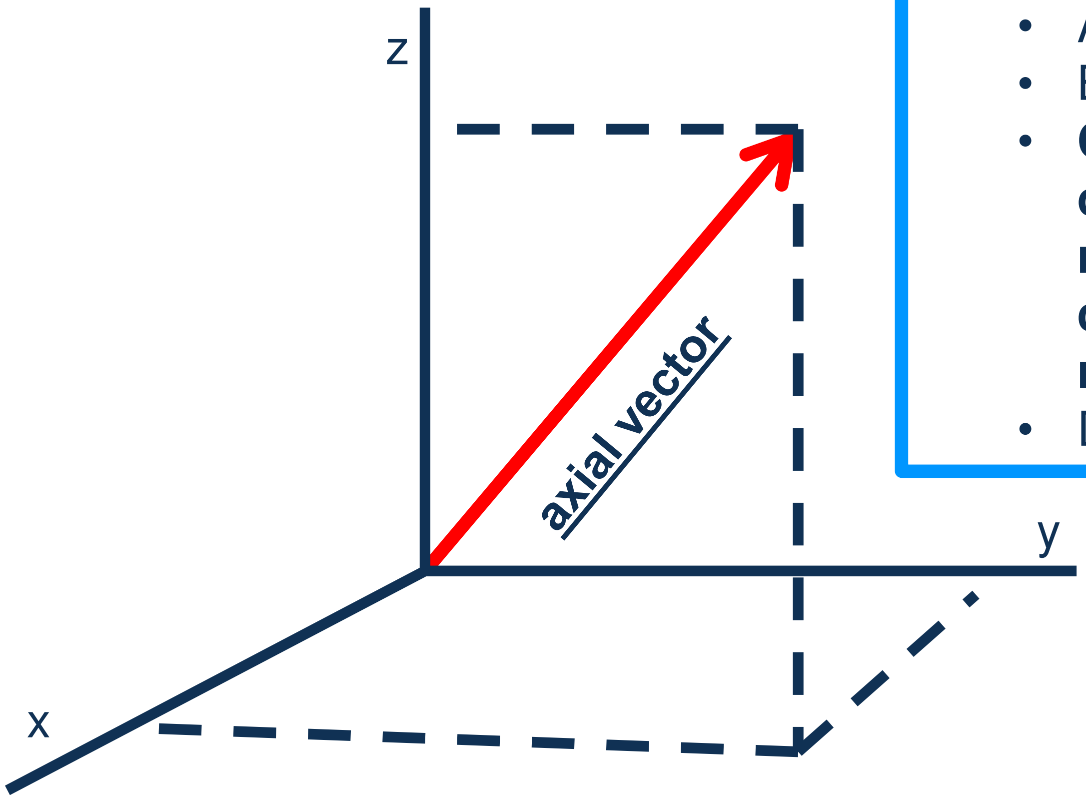
- *Broken* Spherical Symmetry
  - Due to spinning of planet
- => Angular Momentum Dynamics
  - *Not* Constant Angular Momentum
  - Yet (Relatively) Simple Dynamics

Angular Momentum Dynamic Balance

1. Rate of Change of Total Angular Momentum
2. Horizontal Surplus/Deficit of Gravitation in Coordinate System, fixed to the Vortex, hence Rotating with Respect to the Planet.

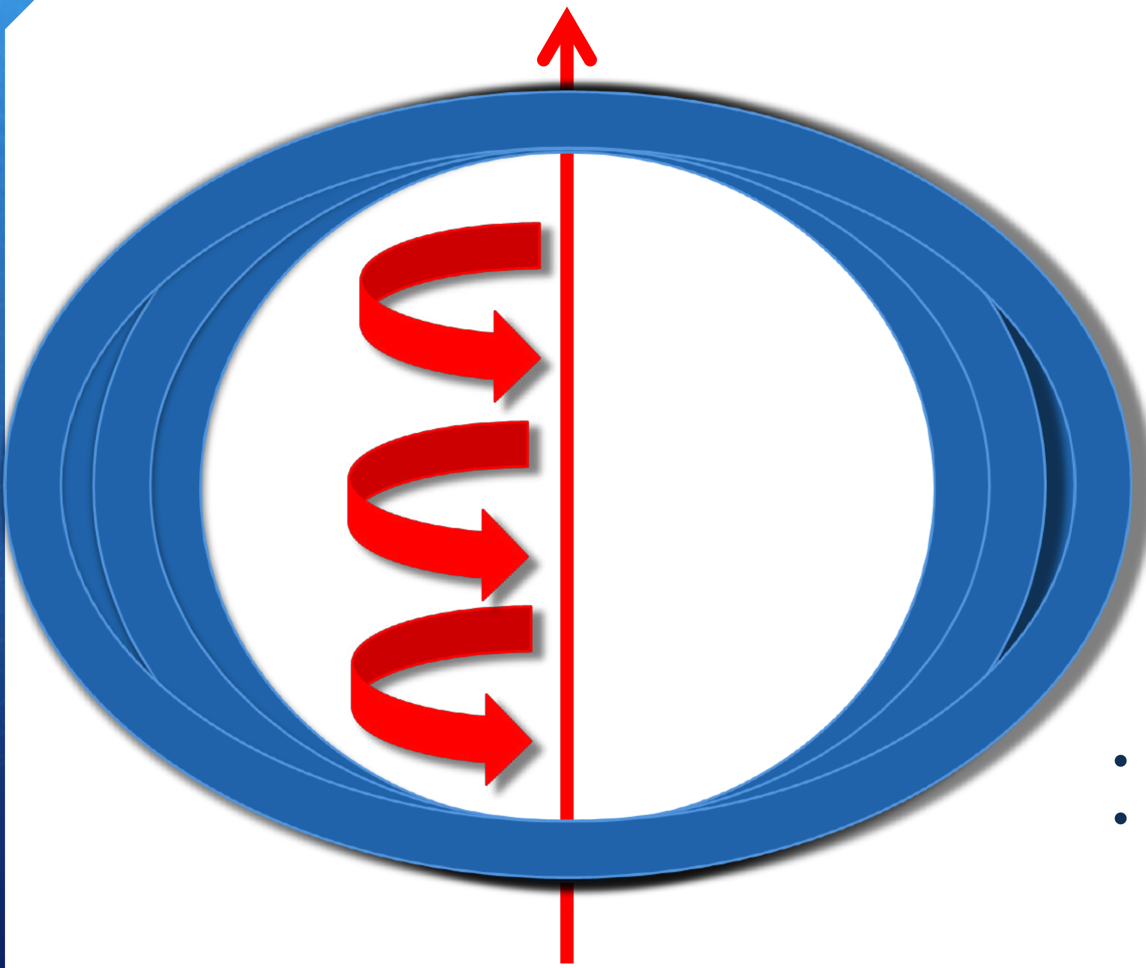
(v.d Toorn & Zimmerman; *Journal of Mathematica Physics*, August 2010)

Dimension of Group of Rotations



- *situation's dimension?*
  - A: 1
  - B: 2
  - **C: 3 is the number of variables (x, y, z) needed to fully characterize any rotation!**
  - D: infinite

Spinning sphere: big centrifuge

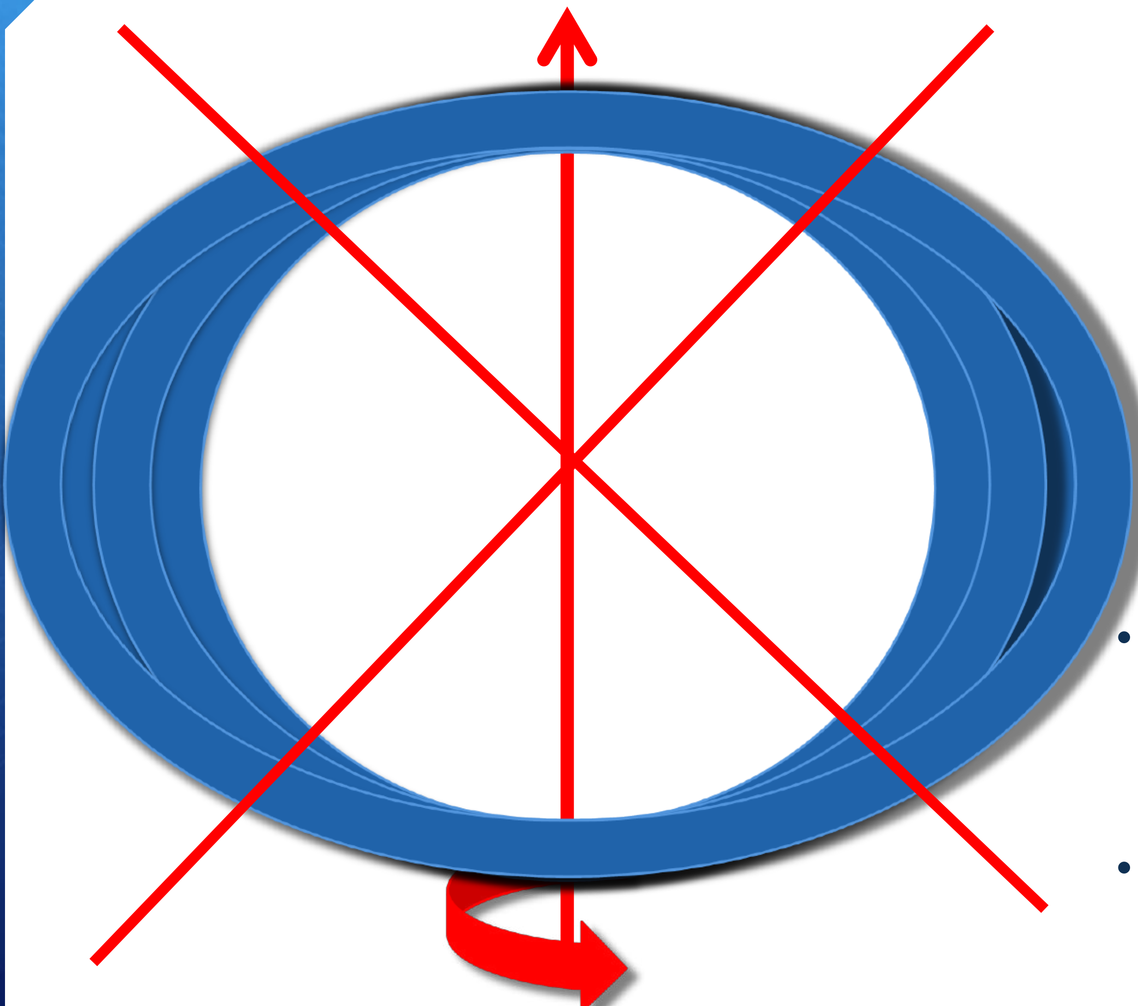


- *situation's dimension?*
  - **A: 1 dominating axis of spin!**
  - B: 2
  - C: 3
  - D: infinite

On a *spinning homogeneous sphere*, the size of the earth, the depth of the oceans would be:

- if 2 km at the poles
- then **13 km at the equator**!

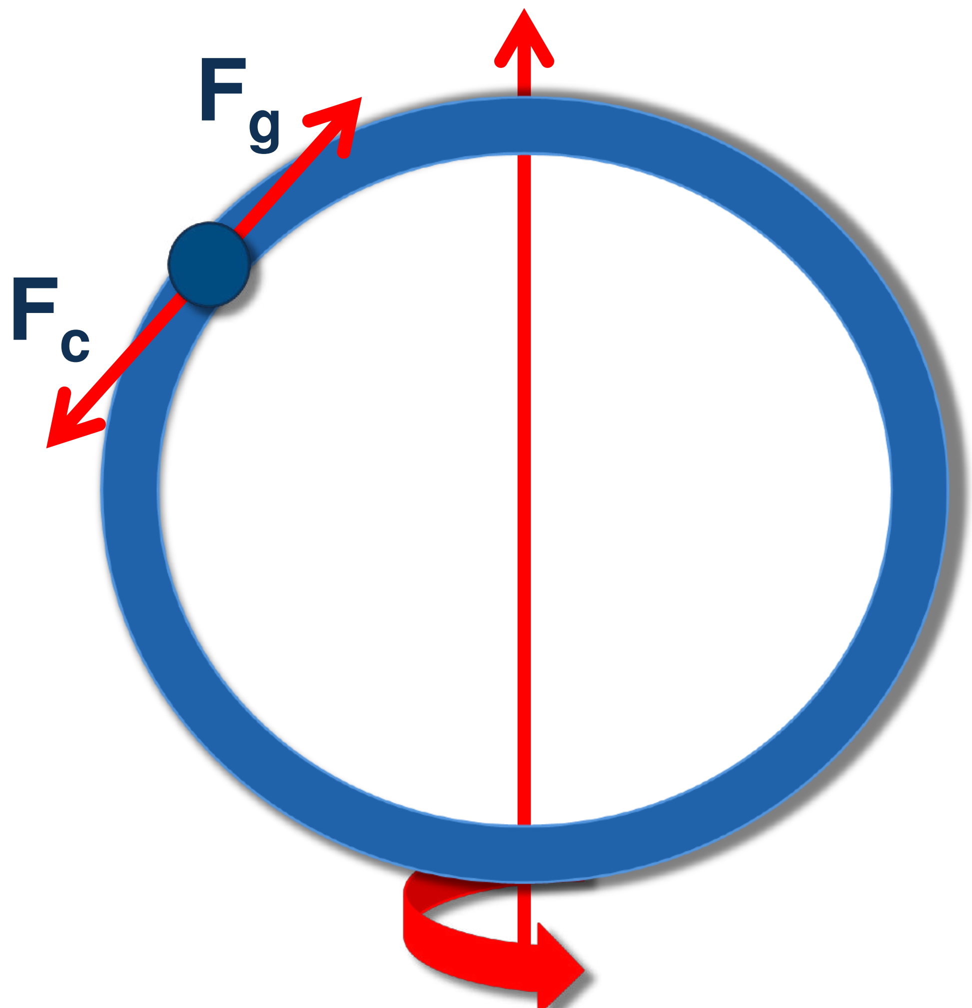
Real Planet: Horizontal Gravitation



In reality, the oceans are approximately of constant depth all over the planet

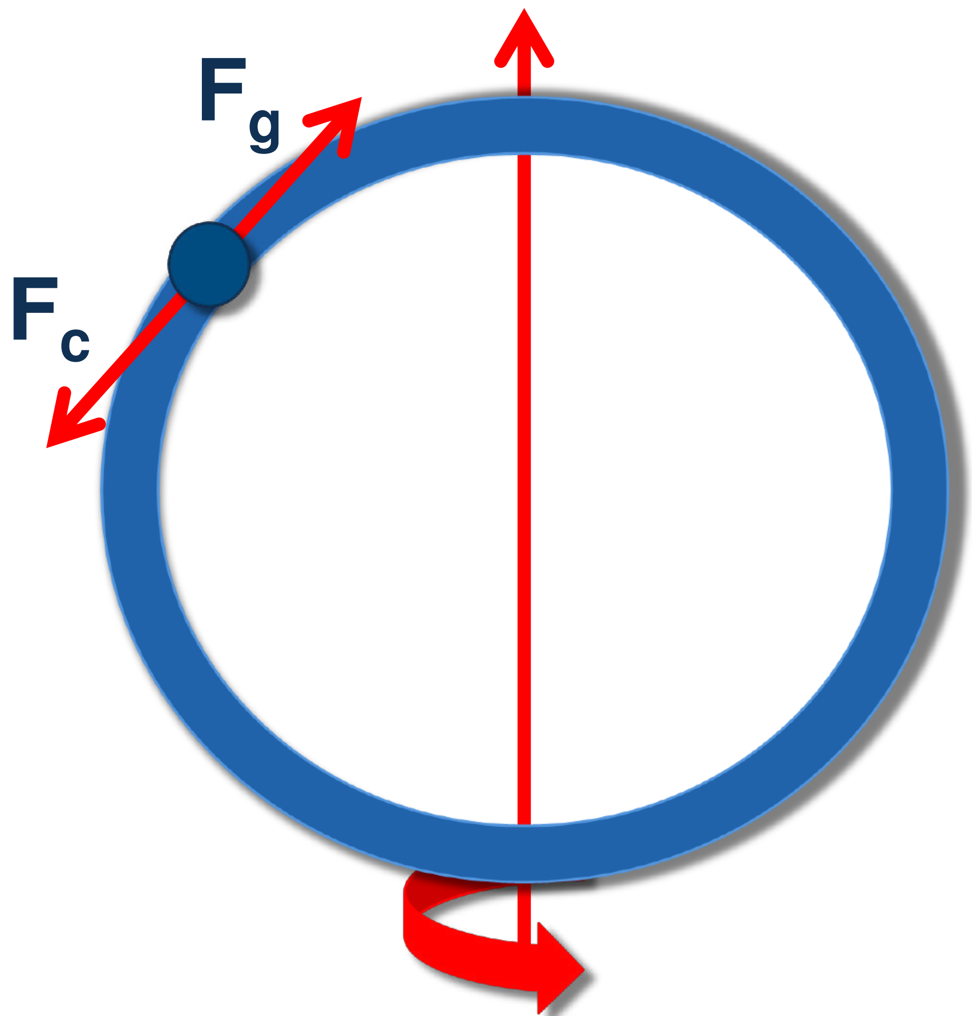
- Centrifugal effect tends to make it considerably deeper at the equator that at poles,
- however, it is compensated...

Horizontal Gravitation



- Ocean is approximately of constant depth all over the planet
- Centrifugal effect tends to make it considerably deeper at the equator that at poles
- Hence centrifugal force must be balanced:
    - By horizontal component of gravitation
    - as induced by deformation of the solid planet.

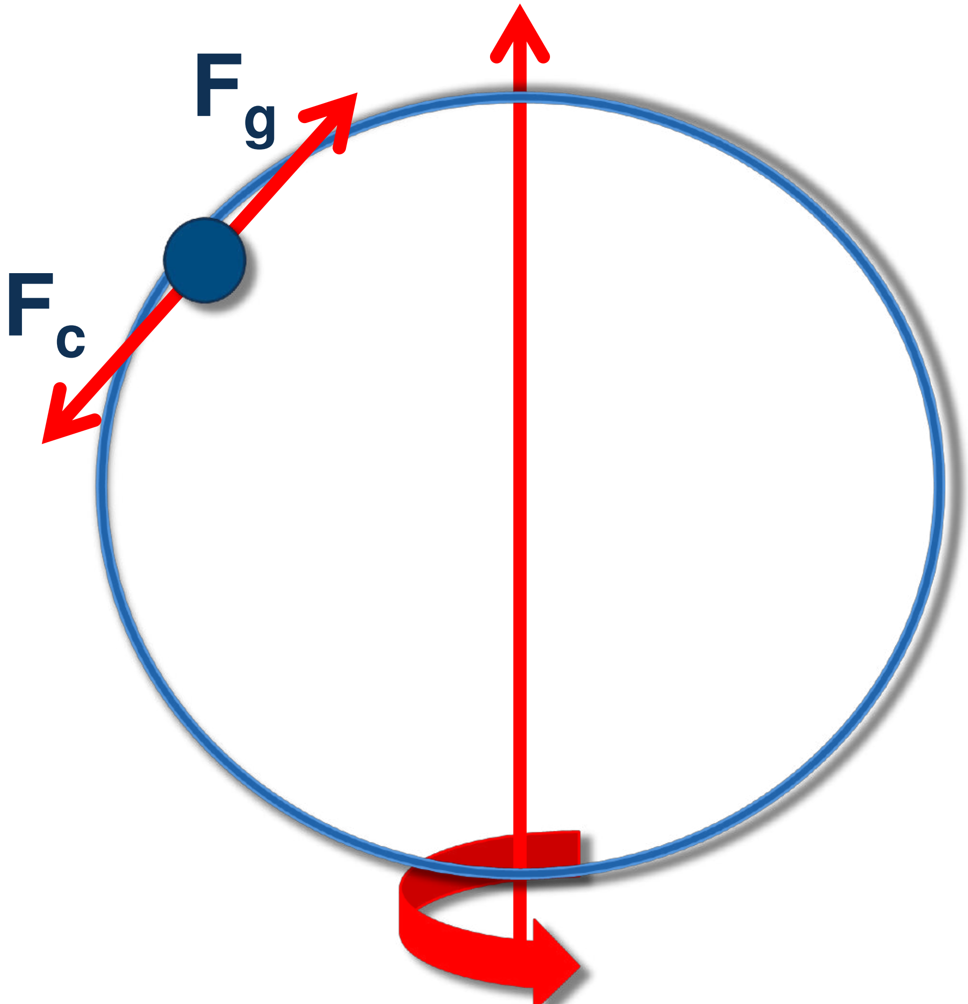
Horizontal Gravitation



- Centrifugal force is balanced:
  - By *horizontal* component of gravitation
  - as induced by a *deformation* of the planet.
  - only a *slight* deformation is sufficient:
  - this deformation is *geometrically* negligible.

(v.d.Toorn & Zimmerman, J.Geophysical Astrophysical Fluid Dynamics, August 2008)

The shallow ocean



- *situation's dimension?*
  - A: 1
  - **B: 2**-dimensional sheet of fluid on a sphere!
  - C: 3
  - D: infinite

- Because the depth of the ocean (~4 km) is small compared to
- vortex size (> 100 km)
- earth radius (~ 6360 km)