The Lowest Vibration Spectra of High-Contrast Composite Structures

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Outline

- Introduction

- Lowest-frequency vibrations of multi-component elastic structures

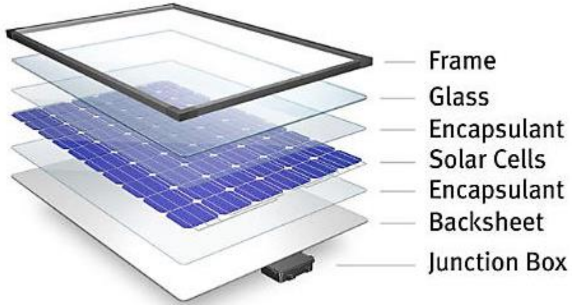
    - Rods
    - Antiplane motion of circular cylinders
    - Plates

- Concluding remarks

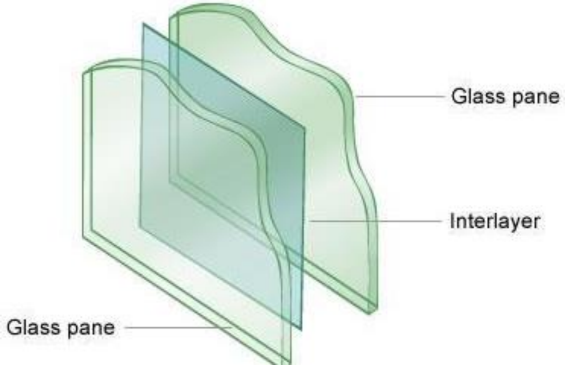
1. Introduction

High-contrast layered structures

- *photovoltaic panels*
- *laminated glass*



www.dupont.com

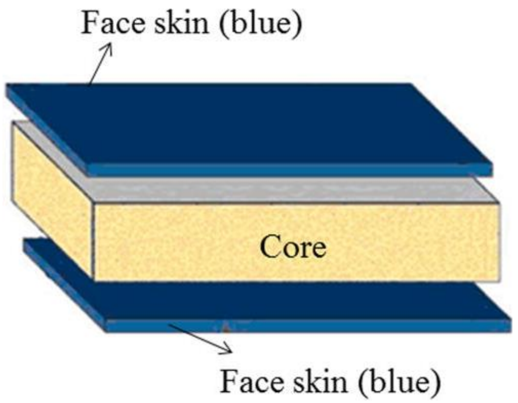


www.gscglass.com

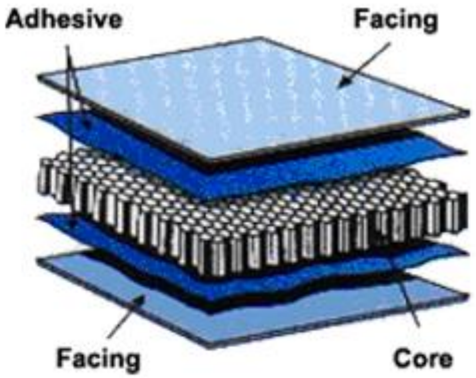
Introduction

Sandwich structures

- ***Classical sandwich plate***



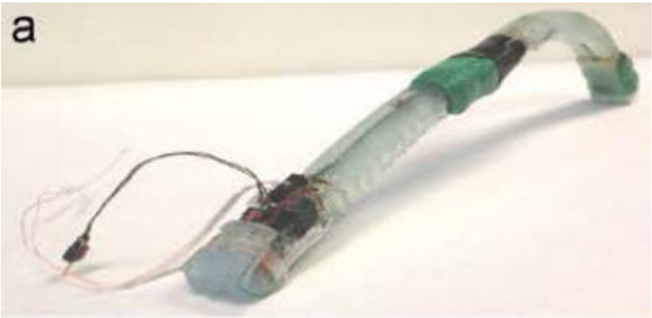
- ***Foam insulation panels***

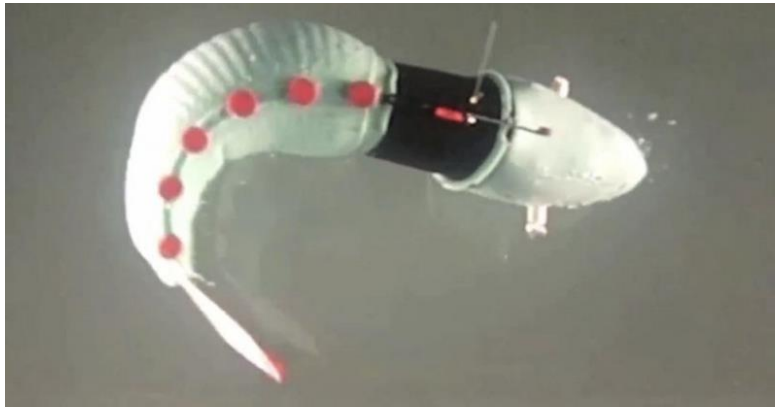


Soft robots

Rus & Tolley, 2015. Design, fabrication and control of soft robots. *Nature*, *521*(7553), 467.

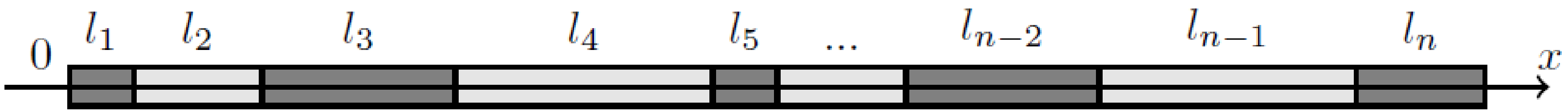
Stokes et al. A hybrid combining hard and soft robots. *Soft Robotics* 1.1 (2014): 70-74





2. Low-frequency vibrations of multi-component high-contrast elastic rods

J. Kaplunov et al., *to appear*



Contrast in

- *Stiffness*
- *Density*
- *Length*

$$\frac{E_i}{E_j}, \qquad \frac{\rho_i}{\rho_j}, \qquad \frac{l_i}{l_j}$$

Small parameters → asymptotic methods

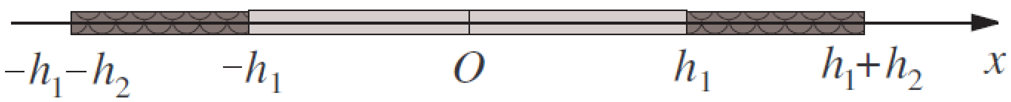
Physical intuition:

Strong components (free ends) - almost rigid body motions

Weak components (fixed b.c.) - almost homogeneous deformations

Toy problem: three-component rod (antisymmetric)

J. Kaplunov et al. *J. Sound Vib.* 366 (2016): 264-276



Equations of motion

$$E_i \frac{d^2 u}{dx^2} + \rho_i \omega^2 u = 0, \qquad i = 1, 2.$$

Free ends

$$u'|_{\pm(h_1+h_2)} = 0.$$

Continuity conditions

$$u|_{\pm(h_1+0)} = u|_{\pm(h_1-0)}, \qquad E_2 u|_{\pm(h_1+0)} = E_1 u|_{\pm(h_1-0)}.$$

Frequency equation

Scaling

$$E = \frac{E_1}{E_2}, \quad \rho = \frac{\rho_1}{\rho_2}, \quad c = \frac{c_1}{c_2}, \quad h = \frac{h_1}{h_2},$$

and

$$\chi = \frac{x}{h_1}, \quad \lambda_i = \frac{\omega h_i}{c_i}, \qquad i = 1, 2.$$

Frequency equation

$$\tan \lambda_1 \tan \lambda_2 = \frac{E}{c}.$$

Low-frequency analysis in view of contrast:

- Global low-frequency behaviour ($\lambda_i \ll 1, \quad i = 1, 2$)
- Local low-frequency behaviour ($\lambda_i \ll 1, \lambda_k \gtrsim 1, \quad i \neq k$)

Global low-frequency behaviour

Conditions on material parameters

$$\lambda_1 \ll 1, \quad \lambda_2 \ll 1 \quad \Rightarrow$$

$$E \ll h \ll \rho^{-1}.$$

Approximate frequency equation

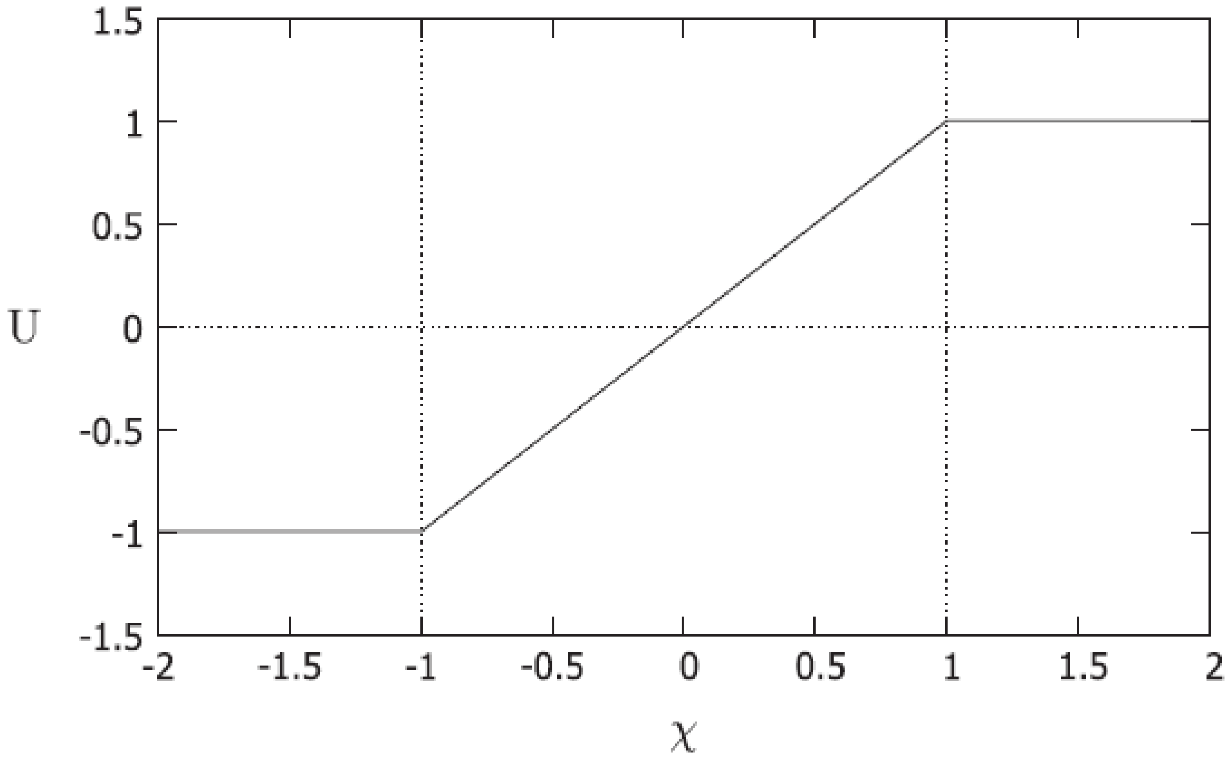
$$\lambda_1 \lambda_2 = \frac{E}{c} \quad \Rightarrow$$

$$\lambda_1 = \sqrt{E\rho}, \quad \lambda_2 = \sqrt{\frac{E}{h}}.$$

Global low-frequency behaviour

Approximate polynomial eigenform

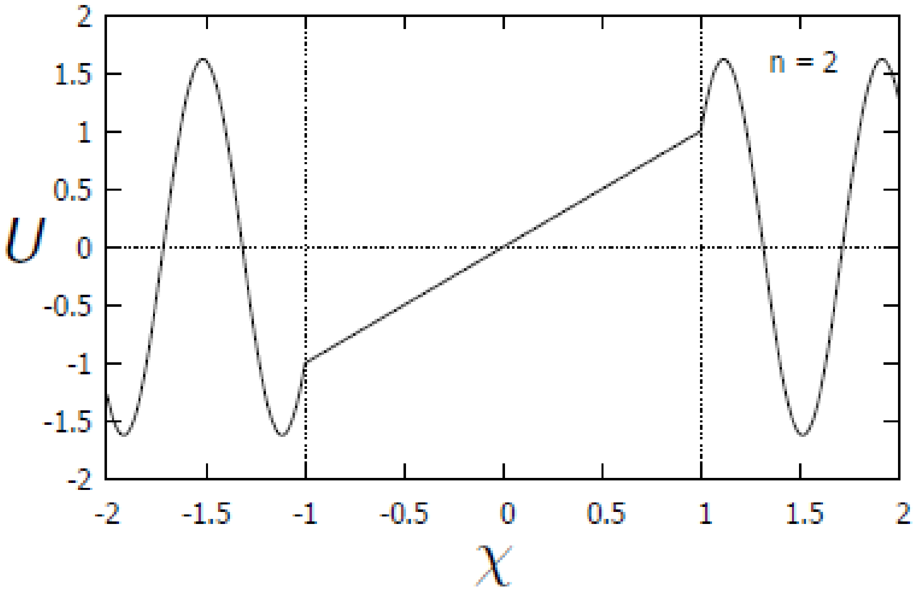
$$U = \begin{cases} 1, & |\chi| > 1; \\ \chi, & |\chi| \le 1. \end{cases}$$

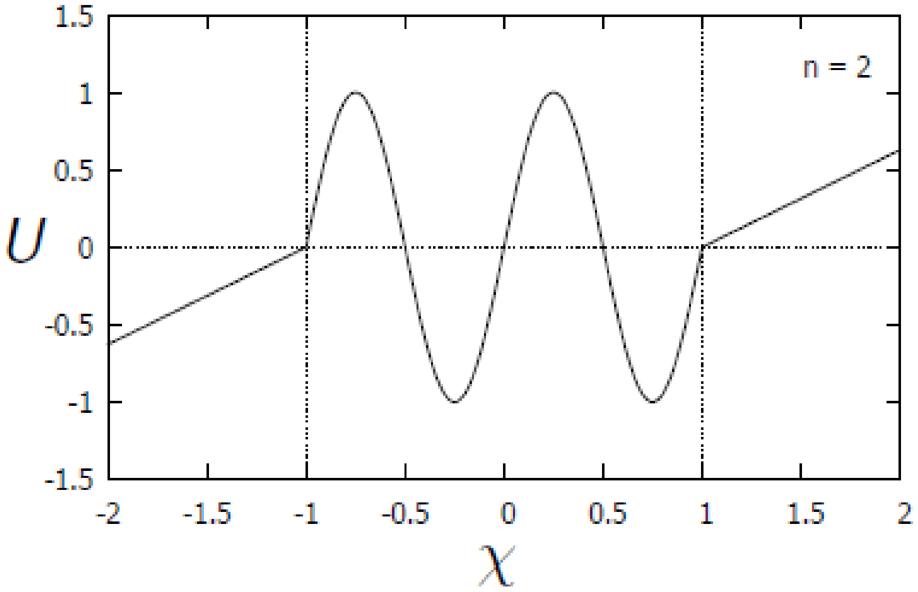


Local low-frequency behaviour

May occur for core or outer sections.

Approximate displacement profiles

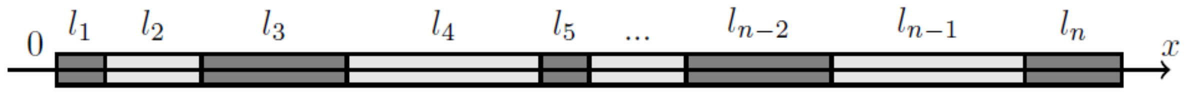




$$n \ll \min\left[\frac{E}{h}, \frac{c}{h}\right]$$

$$\rho h \ll n \ll \frac{h}{c}$$

Multi-component high-contrast elastic rods



Problem parameters

$$\varepsilon = \frac{E_2}{E_1} << 1, \qquad \rho_1 \sim \rho_2, \qquad L_i^j = \frac{l_j}{l_i}, \qquad c_m^2 = \frac{E_m}{\rho_m}.$$

Dimensionless scaling

$$X_i = \frac{x_i}{l_i}, \quad \Omega_i = \frac{\omega l_i}{c_m}, \quad b_i \le X_i \le b_i + 1, \quad b_i = l_i^{-1} \sum_{k=0}^{i-1} l_k, \quad i = \overline{1, n}; \quad m = 1,2.$$

Equations of motion

$$\frac{d^2 u_i}{dX_i^2} + \Omega_i^2 u_i = 0,$$

Boundary conditions

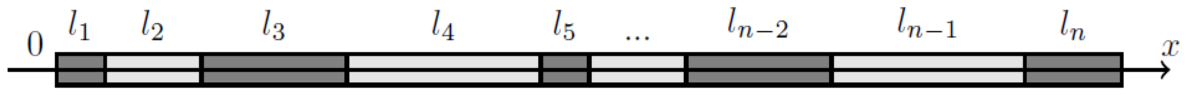
$$\left.\frac{du_1}{dX_1}\right|_{X_1=0} = \left.\frac{du_n}{dX_n}\right|_{X_n=b_n+1} = 0,$$

Continuity

$$\left. u_i \right|_{X_i=b_i+1} = \left. u_{i+1} \right|_{X_{i+1}=b_{i+1}}, \qquad \left.\frac{du_i}{dX_i}\right|_{X_i=b_i+1} = \varepsilon^j L_{i+1}^i \left.\frac{du_{i+1}}{dX_{i+1}}\right|_{X_{i+1}=b_{i+1}}.$$

($j = 1$ or -1 for i^{th} component being stiff or soft, respectively)

Multi-component high-contrast elastic rods



Asymptotic expansions $$u_i = u_{i0} + \varepsilon u_{i1} + \dots,$$

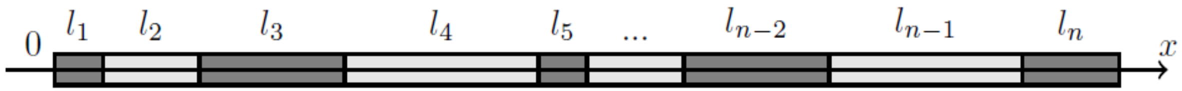
Global low-frequency regime $$\Omega_i^2 \sim \varepsilon, \quad \Omega_i^2 = \varepsilon\left(\Omega_{i0}^2 + \varepsilon\,\Omega_{i1}^2 + \dots\right).$$

- Leading order problem for stiff components

$$\frac{d^2 u_{i0}}{dX_i^2} = 0, \qquad \left.\frac{du_{i0}}{dX_i}\right|_{X_i=b_i} = \left.\frac{du_{i0}}{dX_i}\right|_{X_i=b_i+1} = 0.$$

(almost rigid body motion) $$u_{i0} = C_i = \text{const}$$

Multi-component high-contrast elastic rods



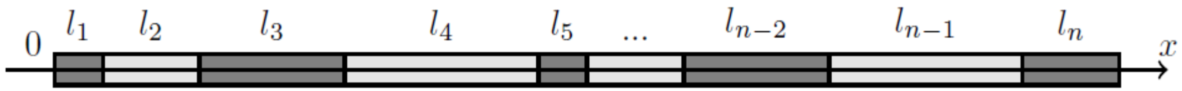
- Leading order problem for soft components

$$\frac{d^2 u_{i0}}{dX_i^2} = 0, \qquad u_i\big|_{X_i=b_i} = C_{i-1}, \quad u_i\big|_{X_i=b_i+1} = C_{i+1}.$$

$$u_{i0} = C_{i-1} + \left(C_{i+1} - C_{i-1}\right)\left(X_i - b_i\right).$$

(almost homogeneous deformation)

Multi-component high-contrast elastic rods



From solvability of next order for stiff components

$$\begin{cases} \Omega_{10}^2 = L_2^1\left(1-\dfrac{C_3}{C_1}\right), \\ \vdots \\ \Omega_{i0}^2 = \left(L_{i-1}^i - L_{i+1}^i\right) - L_{i-1}^i \dfrac{C_{i-2}}{C_i} - L_{i+1}^i \dfrac{C_{i+2}}{C_i}, \\ \vdots \\ \Omega_{n0}^2 = L_{n-1}^n\left(1-\dfrac{C_{n-2}}{C_n}\right); \qquad i = 1,3,5,\ldots,n. \end{cases}$$

Polynomial equation for frequency!

Example: Five-component rod (free ends)

Bicubic frequency equation

$$\Omega_{10}^6 + a_1\Omega_{10}^4 + a_2\Omega_{10}^2 = 0 \quad \Rightarrow \quad \Omega_{10}^2 = kL_2^1, \quad k = 0 \quad \text{or} \quad k = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2L_2^1}.$$

Approximate polynomial eigenform

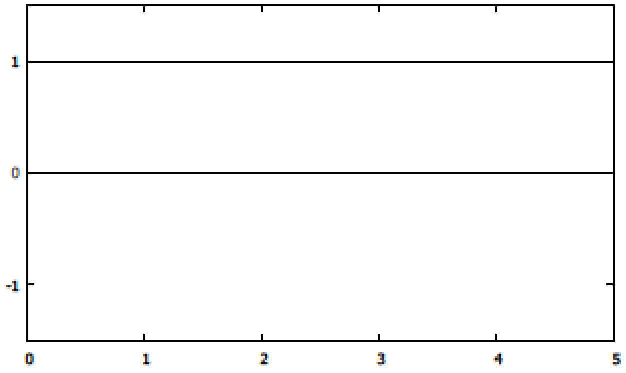
$$u_{10} = 1,$$

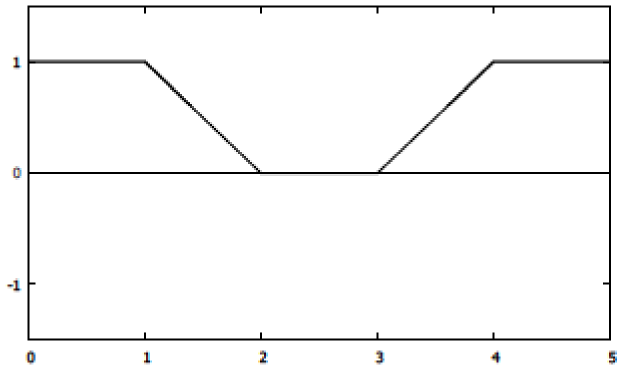
$$u_{20} = 1 - k(X_2 - b_2),$$

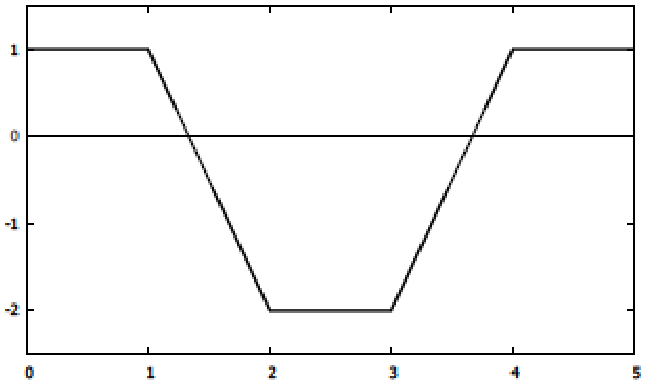
$$u_{30} = 1 - k,$$

$$u_{40} = 1 - k + \frac{L_2^4 L_1^5 k(1-k)}{1 - L_2^4 L_1^5 k}(X_4 - b_4),$$

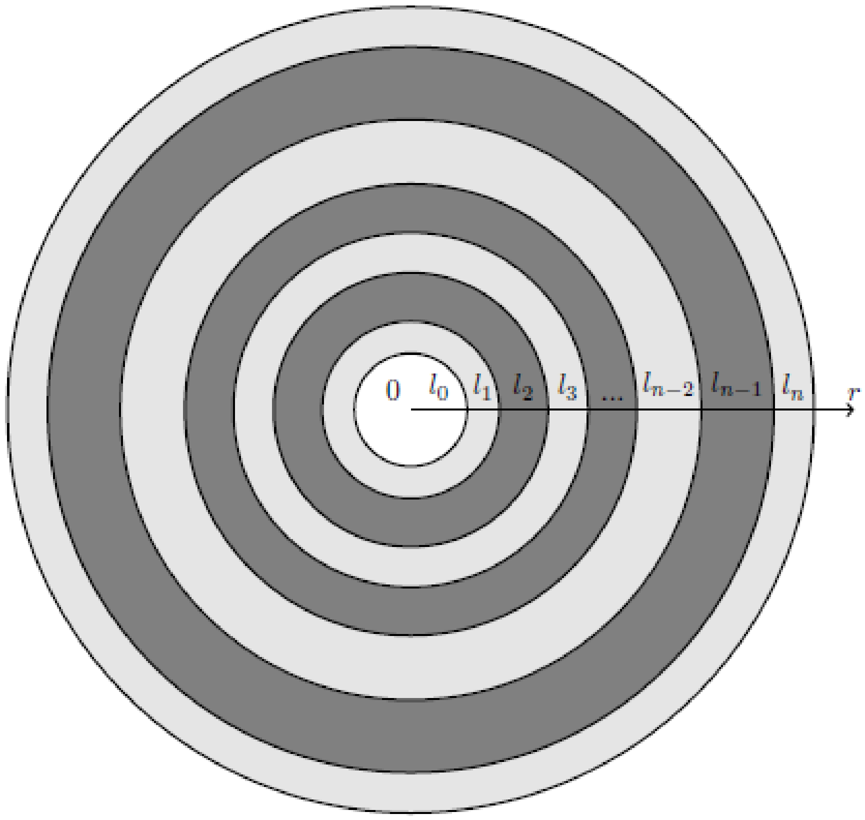
$$u_{50} = \frac{1-k}{1 - L_2^4 L_1^5 k}.$$







Antiplane motion of concentric circular cylinders



Problem parameters

$$\varepsilon = \frac{\mu_1}{\mu_2} << 1, \quad \rho_1 \sim \rho_2, \quad L_i^j = \frac{l_j}{l_i}, \quad c_m^2 = \frac{\mu_m}{\rho_m}, \quad m = 1,2.$$

$$R_i = \frac{r_i}{l_i}, \quad \Omega_i = \frac{\omega l_i}{c_m}, \quad b_i \le R_i \le b_i + 1, \quad b_i = l_i^{-1} \sum_{k=0}^{i-1} l_k, \quad i = \overline{1, n}.$$

Equations of motion

$$R_i \frac{d^2 u_i}{dR_i^2} + \frac{du_i}{dR_i} + R_i \Omega_i^2 u_i = 0.$$

Boundary conditions

$$u_1\big|_{R_1 = b_1} = u_n\big|_{R_n = b_n + 1} = 0.$$

Continuity

$$u_i\big|_{R_i = b_i + 1} = u_{i+1}\big|_{R_{i+1} = b_{i+1}}, \quad \left.\frac{du_i}{dR_i}\right|_{R_i = b_i + 1} = \varepsilon^j L_{i+1}^i \left.\frac{du_{i+1}}{dR_{i+1}}\right|_{R_{i+1} = b_{i+1}}.$$

($j = 1$ or -1 for i^{th} component being stiff or soft, respectively)

Antiplane motion of concentric circular cylinders

Summary of the approach

- Global low-frequency perturbation

↓

- Rigid body motions of stiffer components
  (at leading order)

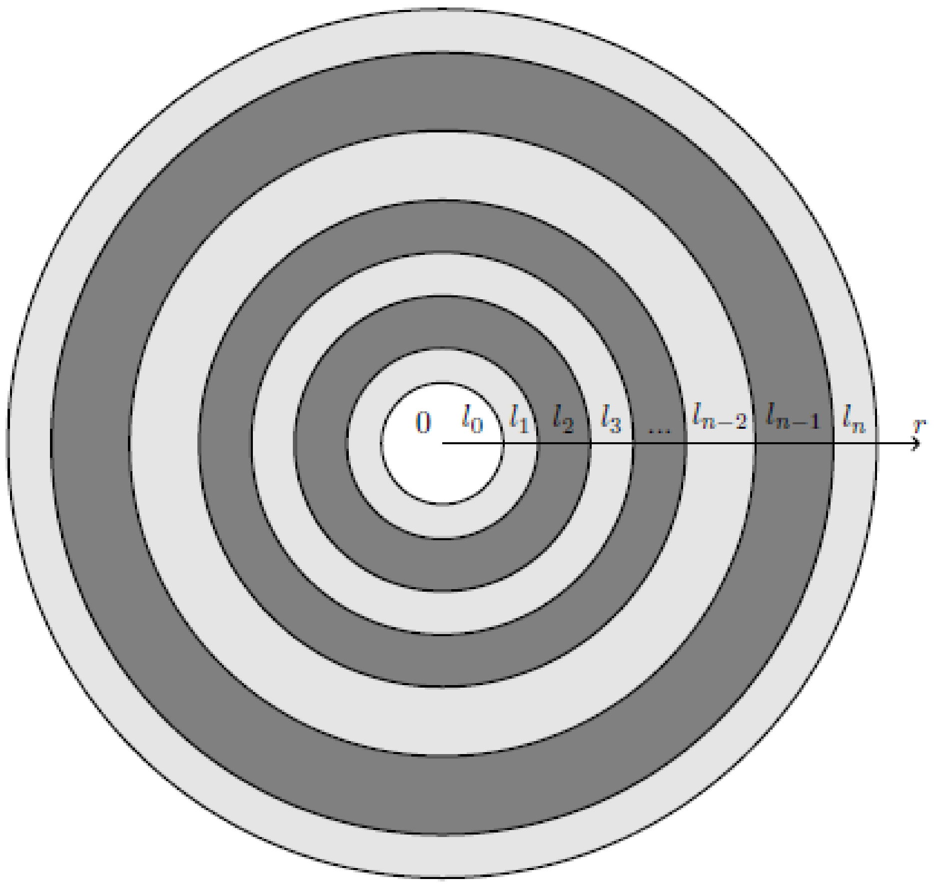
↓

- Leading order solution for softer components, involving logarithmic functions

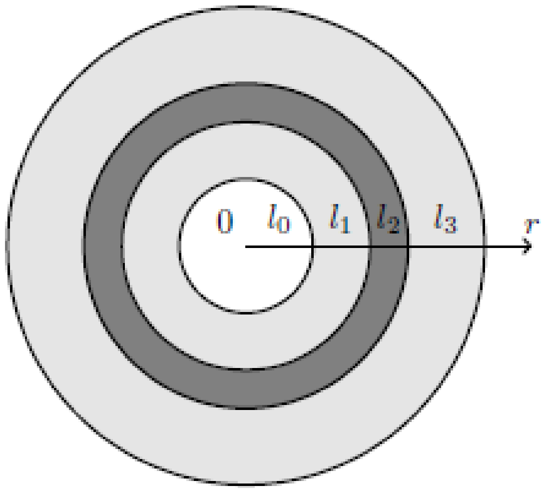
↓

- Solvability of the next order problem
  for stiffer components

→ Polynomial equation for frequency



Example: Three-layered cylinder



Frequency

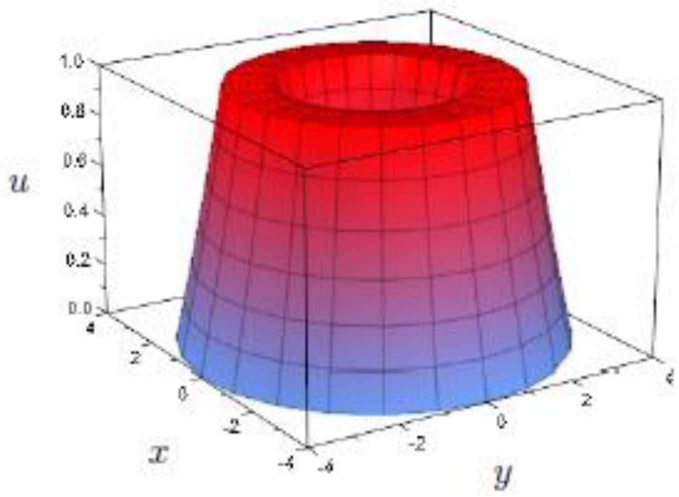
$$\Omega_{20}^2 = \frac{2(\delta_1 + \delta_3)}{2b_2 + 1}, \qquad \delta_i = \frac{1}{\ln\left(1 + b_i^{-1}\right)}.$$

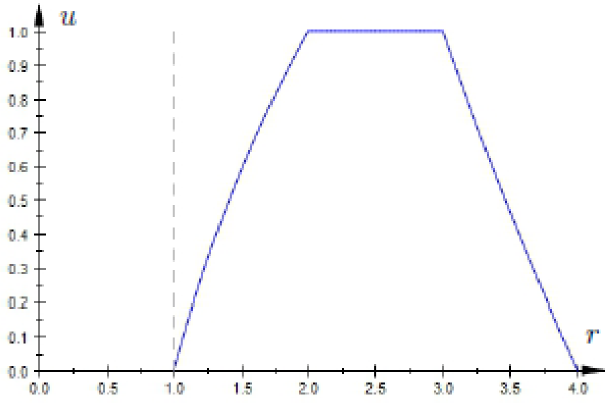
Eigenform

$$u_{10} = \delta_1 \ln\left(\frac{R_1}{b_1}\right),$$

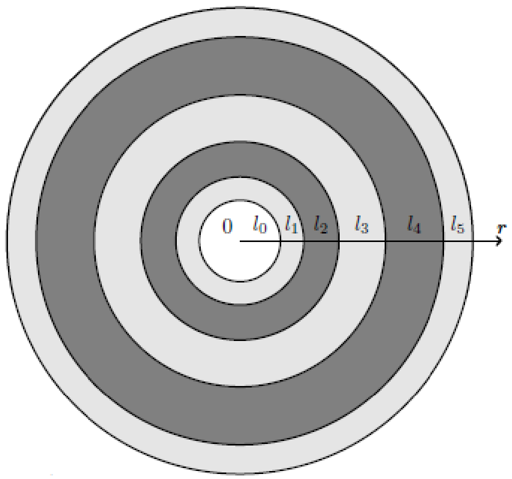
$$u_{20} = 1,$$

$$u_{30} = \delta_3 \ln\left(\frac{b_3 + 1}{R_3}\right).$$





Example: Five-layered cylinder



Frequency $$\Omega_{20}^2 = \frac{2}{2b_2+1}\left(\delta_1 + (1-k)\delta_3\right)$$ (two roots for k)

Eigenform

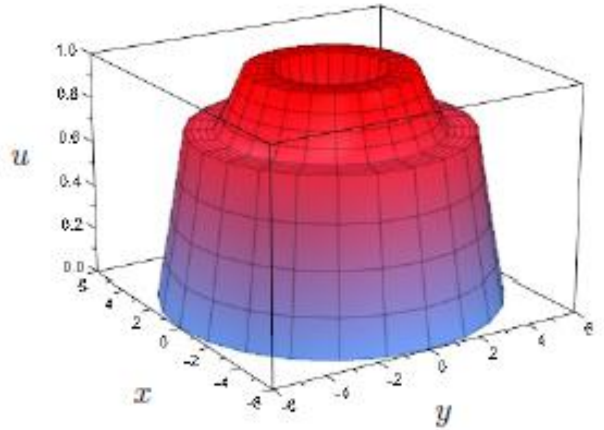
$$u_{10} = \delta_1 \ln\left(\frac{R_1}{b_1}\right),$$

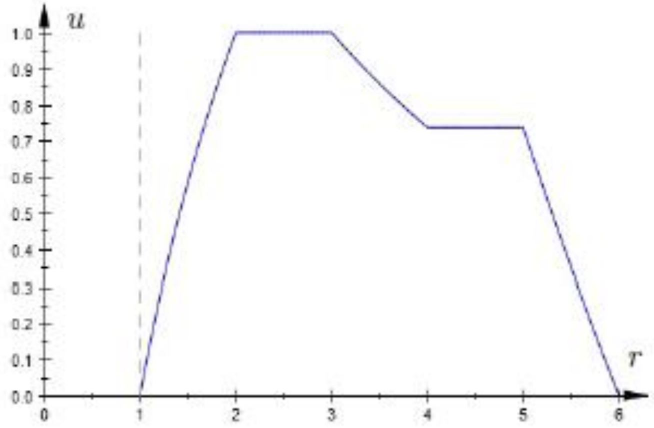
$$u_{20} = 1,$$

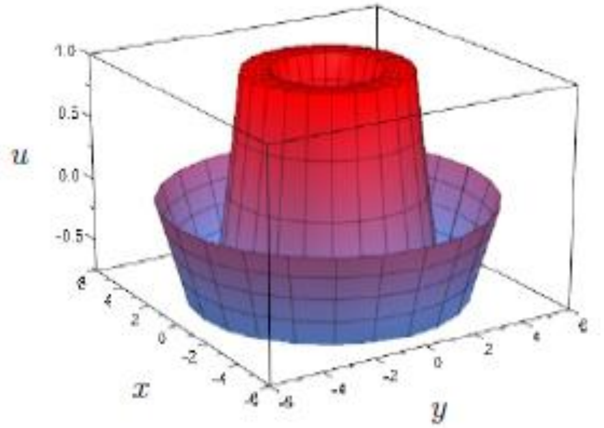
$$u_{30} = -\delta_3 \left\{ \ln\left(\frac{R_3}{b_3+1}\right) - k \ln\left(\frac{R_3}{b_3}\right) \right\},$$

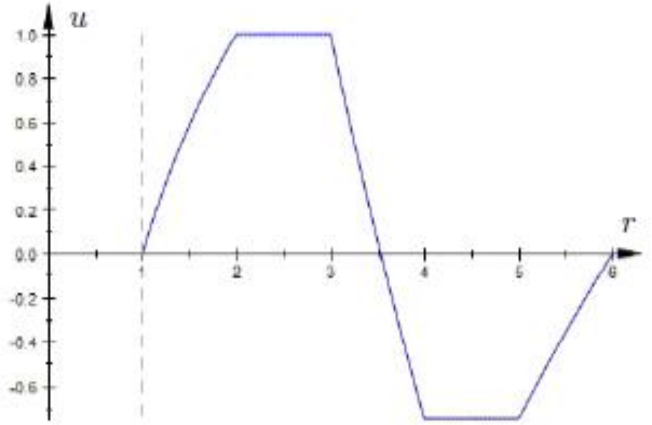
$$u_{40} = k,$$

$$u_{50} = k\delta_5 \ln\left(\frac{b_5+1}{R_5}\right).$$









Low-frequency vibrations of high-contrast three-layered plates (antisymmetric)

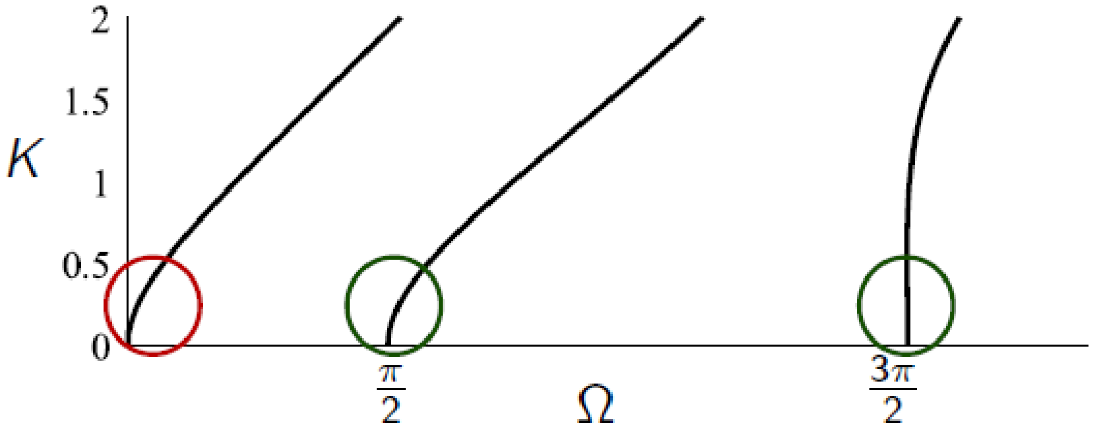
Preliminary remarks

- Rayleigh-Lamb dispersion relation for a single-layered plate

$$\gamma^4 \frac{\sinh \alpha}{\alpha} \cosh \beta - \beta^2 K^2 \cosh \alpha \frac{\sinh \beta}{\beta} = 0,$$

$$\alpha^2 = K^2 - \varkappa^2 \Omega^2, \quad \beta^2 = K^2 - \Omega^2, \quad \gamma^2 = K^2 - \frac{1}{2}\Omega^2, \quad \varkappa = \frac{c_2}{c_1}$$

$$K = kh, \quad \Omega = \frac{\omega h}{c_2}$$



NO CHANCE OF TWO-MODE APPROXIMATIONS!

Low-frequency ($\Omega \ll 1$)

At the leading order $\Omega^2 \sim K^4$

High-frequency approximations near cut-off frequencies $\Omega_* \sim 1$ ($|\Omega - \Omega_*| \ll 1$)

At the leading order $K^2 \sim \Omega^2 - \Omega_*^2$

Composite (non-uniformly asymptotic) plate theories

Originate from Timoshenko-Reissner-Mindlin ad hoc theories.

$$\overbrace{D_a \frac{d^4 W}{d\xi^4} - \Omega^2 W}^{\text{low-frequency}} \underbrace{\Omega^2 W + B_a \Omega^2 \frac{d^2 W}{d\xi^2} + C_a \Omega^4 W}_{\text{high-frequency}} = 0,$$

Contributions for composite plate and shells theories

V.L. Berdichevsky. Variational principles of continuum mechanics: I. Fundamentals. Springer Science and Business Media, 2009

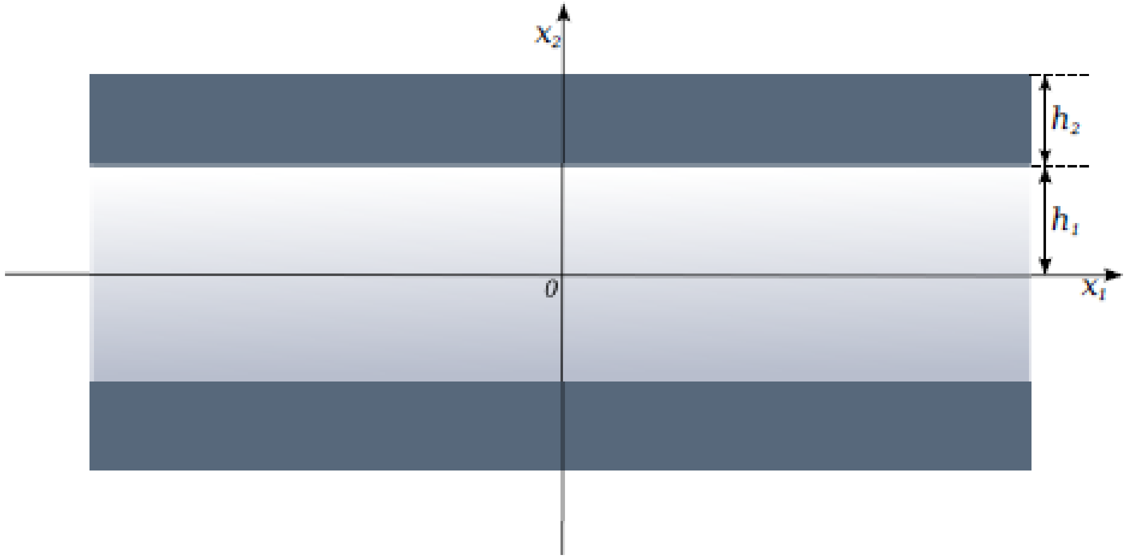
K.C. Le. Vibrations of shells and rods. Springer Science and Business Media, 2012

I.V. Andrianov, J. Awrejcewicz, L.I. Manevitch. Asymptotical mechanics of thin-walled structures. Springer Science and Business Media, 2013

Low-frequency vibrations of high-contrast three-layered plates

Kaplunov et al. *Int. J. Solids Struct.* 113 (2017): 169-179

Statement of the problem



Equations of motion

$$\sigma_{ji,j} = \rho \ddot{u}_i, \quad i = 1,2 \quad \text{for layers I and II}$$

Boundary and continuity conditions

$$\sigma_{12}^{\mathrm{II}} = 0, \quad \sigma_{22}^{\mathrm{II}} = 0 \quad \text{at} \quad x_2 = h_1 + h_2$$

$$\sigma_{12}^{\mathrm{I}} = \sigma_{12}^{\mathrm{II}}, \quad \sigma_{22}^{\mathrm{I}} = \sigma_{22}^{\mathrm{II}} \quad \text{and} \quad u_1^{\mathrm{I}} = u_1^{\mathrm{II}}, \quad u_2^{\mathrm{I}} = u_2^{\mathrm{II}} \quad \text{at} \quad x_2 = h_1$$

Dispersion relation

Ustinov, Doklady Physics (1976); Lee, Chang, Journal of Elasticity (1979)

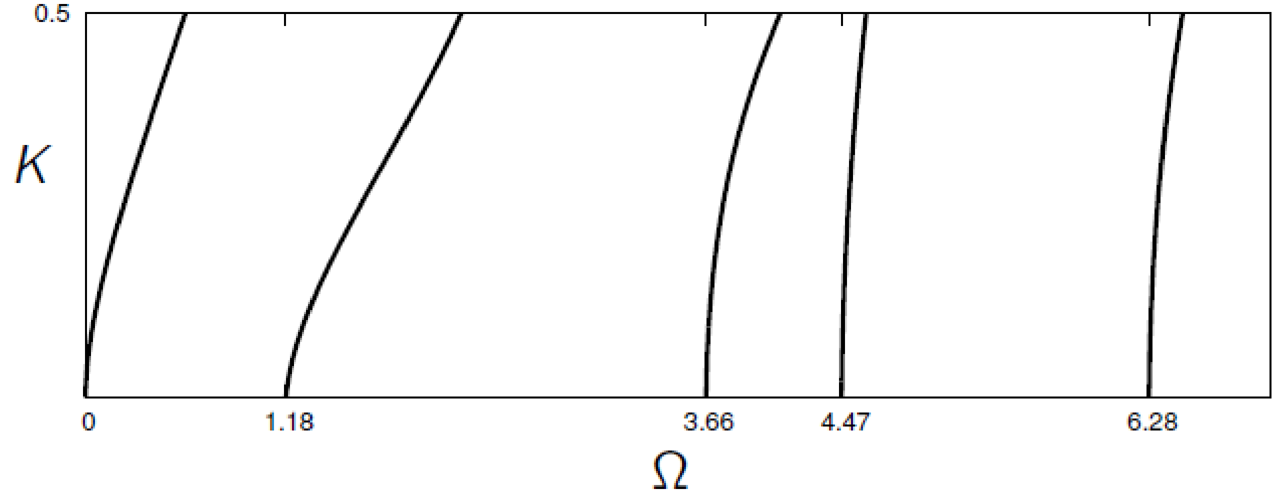
$$
\begin{aligned}
&4K^2h^3\alpha_2\beta_2F_4\left[F_1F_2C_{\beta_1}S_{\alpha_1}-2\alpha_1\beta_1(\varepsilon-1)F_3C_{\alpha_1}S_{\beta_1}\right]+\\
&h\alpha_2\beta_2C_{\alpha_2}C_{\beta_2}\left[4\alpha_1\beta_1K^2\left(h^4F_3^2+{F_4}^2(\varepsilon-1)^2\right)C_{\alpha_1}S_{\beta_1}-\right.\\
&\qquad\qquad\qquad\left.\left(4K^4h^4F_2^2+{F_4}^2F_1^2\right)S_{\alpha_1}C_{\beta_1}\right]+\\
&C_{\beta_2}S_{\alpha_2}\varepsilon\beta_2(\beta_2^2-K^2h^2)(\beta_1^2-K^2)\left[4\alpha_2^2\beta_1K^2h^2S_{\alpha_1}S_{\beta_1}-{F_4}^2\alpha_1C_{\alpha_1}C_{\beta_1}\right]+\\
&C_{\alpha_2}S_{\beta_2}\varepsilon\alpha_2(\beta_2^2-K^2h^2)(\beta_1^2-K^2)\left[4\alpha_1\beta_2^2K^2h^2C_{\alpha_1}C_{\beta_1}-{F_4}^2\beta_1S_{\alpha_1}S_{\beta_1}\right]+\\
&h^3S_{\alpha_2}S_{\beta_2}\left[\left(4\alpha_2^2\beta_2^2K^2F_1^2+K^2{F_4}^2F_2^2\right)C_{\beta_1}S_{\alpha_1}-\right.\\
&\qquad\qquad\left.\alpha_1\beta_1\left(16\alpha_2^2\beta_2^2(\varepsilon-1)^2K^4+{F_4}^2F_3^2\right)C_{\alpha_1}S_{\beta_1}\right]=0
\end{aligned}
$$

$\Omega = \dfrac{\omega h_1}{c_2^{\mathrm{I}}}$, $\quad K = kh_1$, $\qquad C_{\alpha_j}, C_{\beta_j}, S_{\alpha_j}, S_{\beta_j}$ - hyperbolic functions

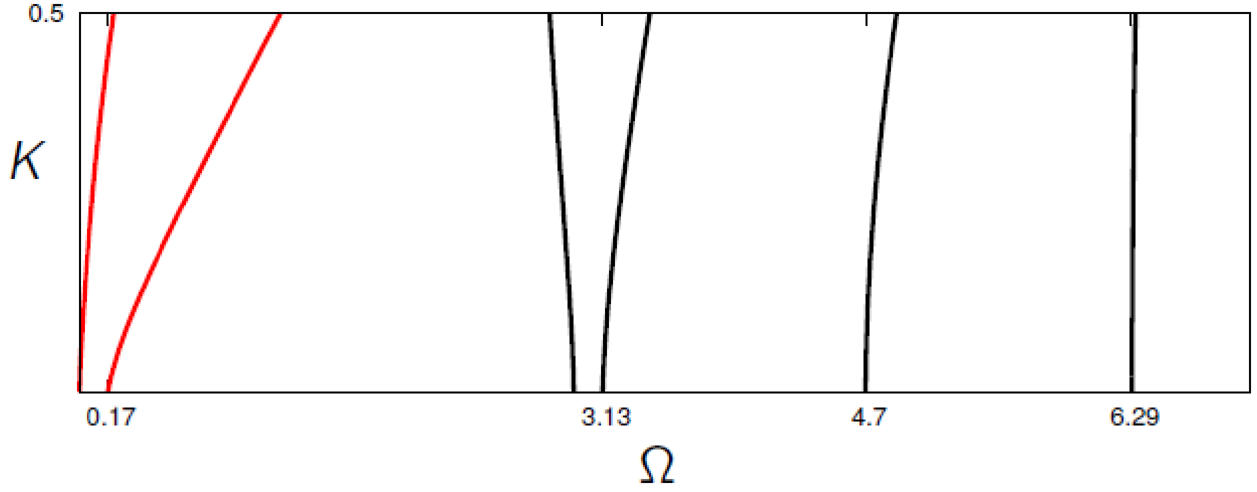
F_i, $\quad i = 1..4$, $\qquad \alpha_j, \beta_j$, $\quad j = 1, 2$ - functions of Ω and K, $\quad \varepsilon = \dfrac{\mu_1}{\mu_2}$

Dispersion curves

No contrast



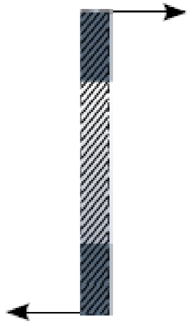
Effect of contrast

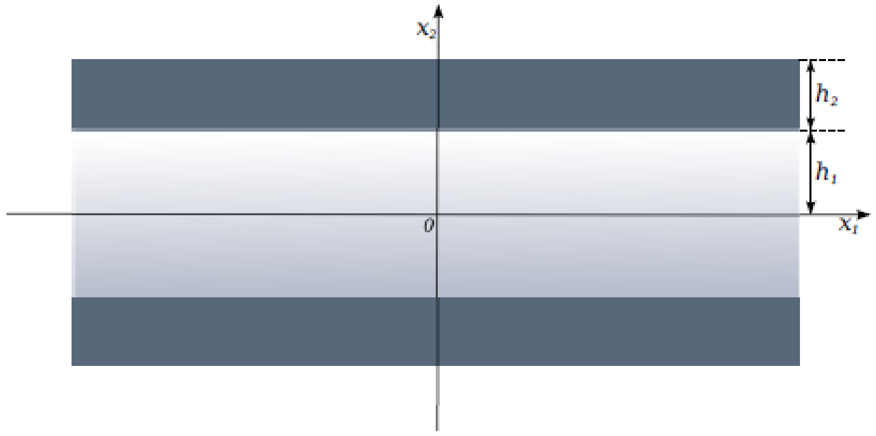


NEED OF TWO-MODE MODELS!

1D eigenvalue problem for shear cut-off

Flexural motion $\frac{\partial}{\partial x_1} = 0, \qquad u_2 = 0$





Frequency equation

$$\tan(\Omega)\tan\left(\sqrt{\frac{\varepsilon}{r}}h\,\Omega\right) = \sqrt{\varepsilon r}$$

Condition for a first shear cut-off frequency to be small

$$\boxed{r \ll h \ll \varepsilon^{-1}}$$

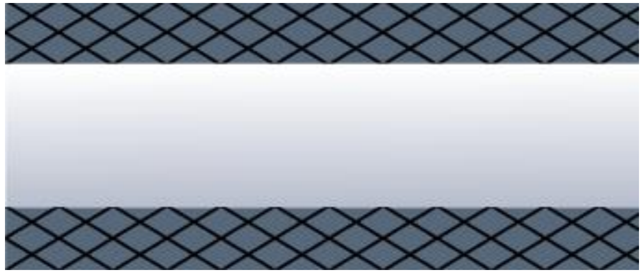
Frequency $\Omega^2 \sim \frac{r}{h}$

where $r = \frac{\rho_1}{\rho_2}, \quad h = \frac{h_2}{h_1}, \quad \varepsilon = \frac{\mu_1}{\mu_2}$

Some three-layered structures satisfying the condition $r \ll h \ll \varepsilon^{-1}$

A) Photovoltaic panels

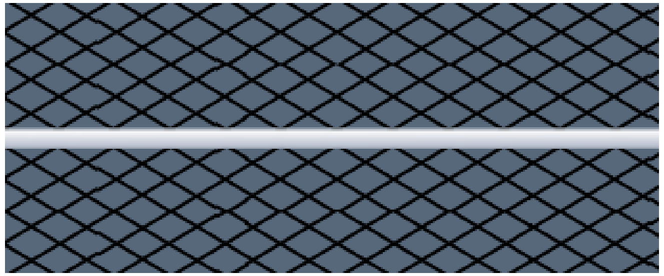
$\varepsilon << 1,\ h \sim 1,\ \rho \sim \varepsilon$



(stiff skin layers and light core layer)

B) Laminated glass

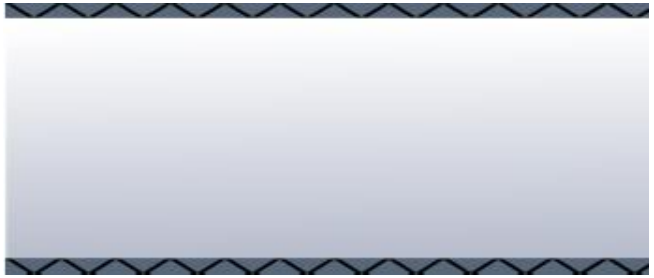
$\varepsilon << 1,\ h \sim \varepsilon^{-1/4},\ \rho \sim 1$



(stiff skin layers and light thin core layer)

C) Sandwich structure

$\varepsilon << 1,\ h \sim \varepsilon,\ \rho \sim \varepsilon^2$



(stiff thin skin layers and light core layer)

UNEXPECTEDLY LOW FIRST SHEAR CUT-OFF FREQUENCIES!

Long-wave low-frequency asymptotic approximation of the dispersion relation

For $K \ll 1$ and $\Omega \ll 1$

$$\gamma_1\Omega^2 + \gamma_2 K^4 + \gamma_3 K^2\Omega^2 + \gamma_4 K^6 + \gamma_5\Omega^4 + \gamma_6 K^4\Omega^2 + \gamma_7 K^8 + \gamma_8 K^2\Omega^4 + \gamma_9 K^2\Omega^6 + \gamma_{10}\Omega^6 + \ldots = 0$$

Multi-parametric analysis

$$\varepsilon \ll 1, \quad h \sim \varepsilon^a, \quad r \sim \varepsilon^b$$

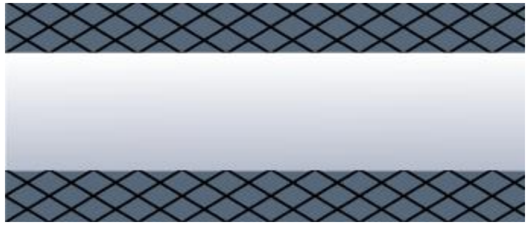
Expanding coefficients

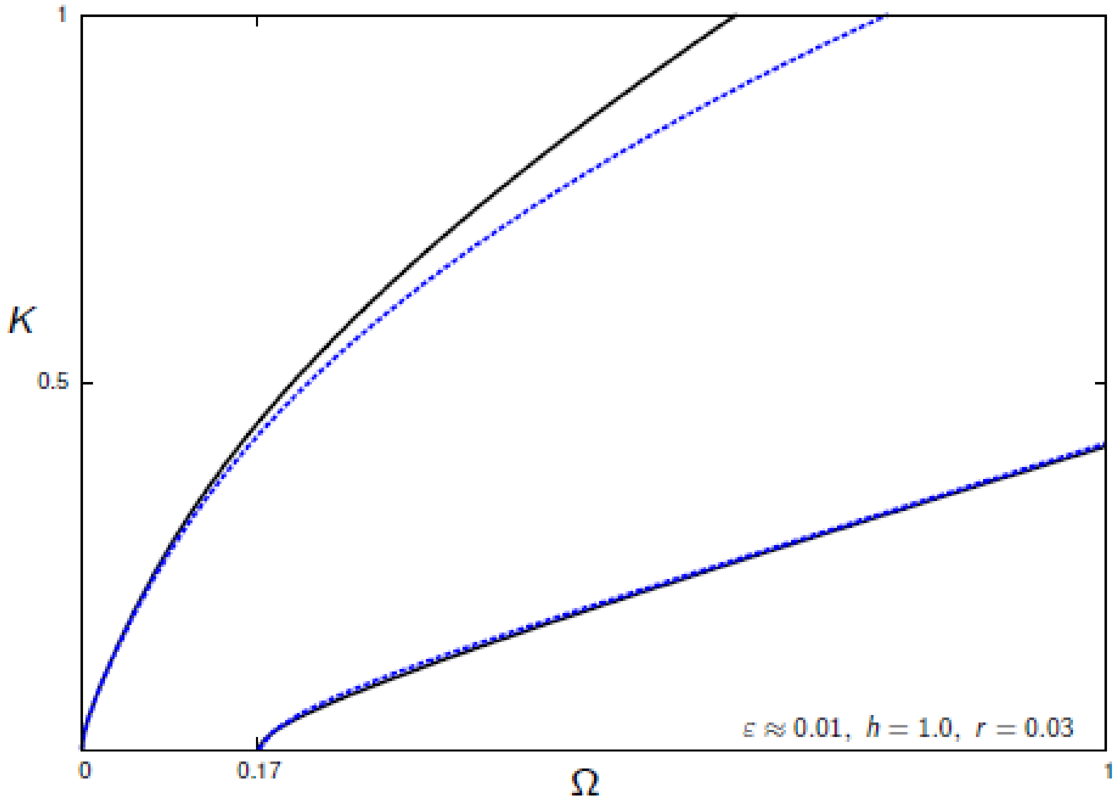
$$\gamma_i \to G_i \varepsilon^c$$

Low-frequency dispersion behaviour

A) Photovoltaic panels $\varepsilon \ll 1, \quad h \sim 1, \quad r \sim \varepsilon$

(stiff skin layers and light core layer)





Retain leading order terms for both:

(i) fundamental mode ($\Omega \sim K^2$)

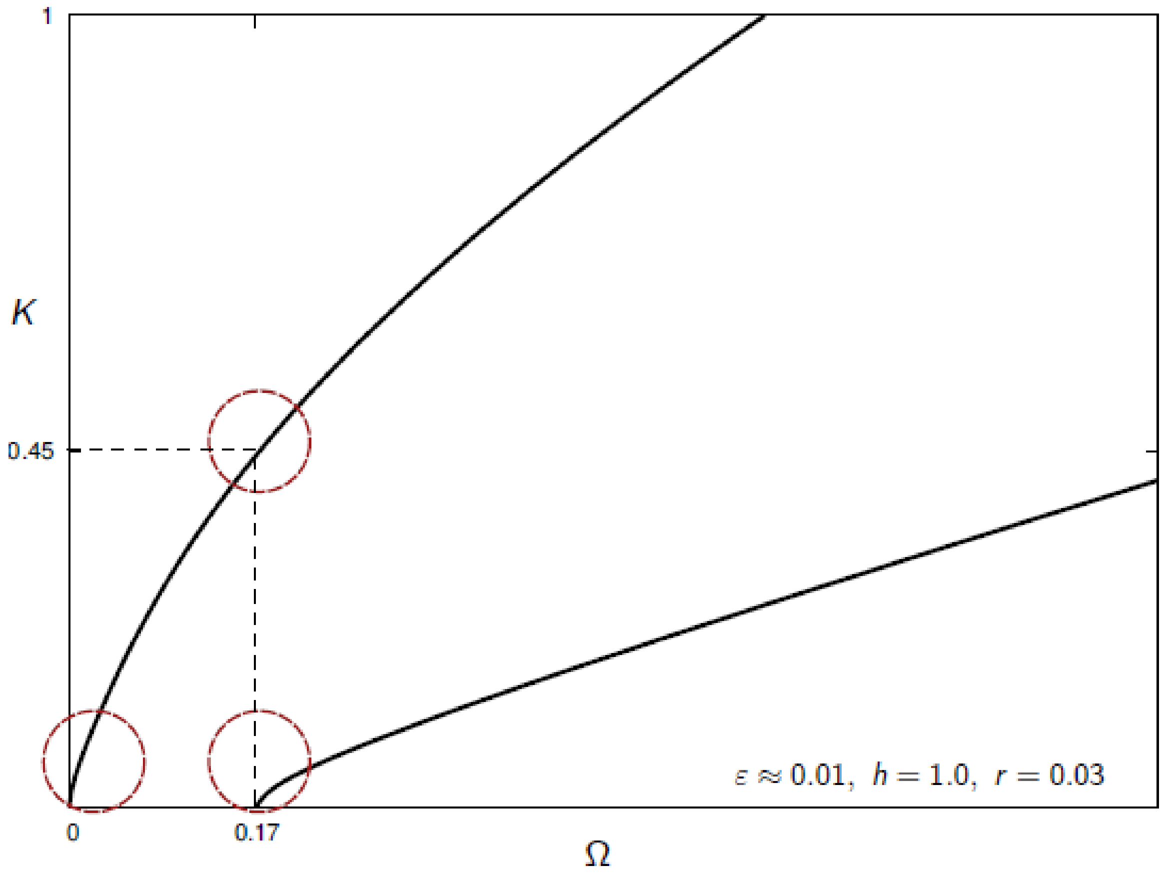
(ii) shear mode cut-off ($\Omega_{sh} \sim \sqrt{\varepsilon}$)

UNIFORM TWO-MODE APPROXIMATIONS

$$G_1 \varepsilon \Omega^2 + G_2 \varepsilon K^4 + G_3 K^2 \Omega^2 + G_4 K^6 + G_5 \Omega^4 = 0$$

Local approximations

Three local approximations

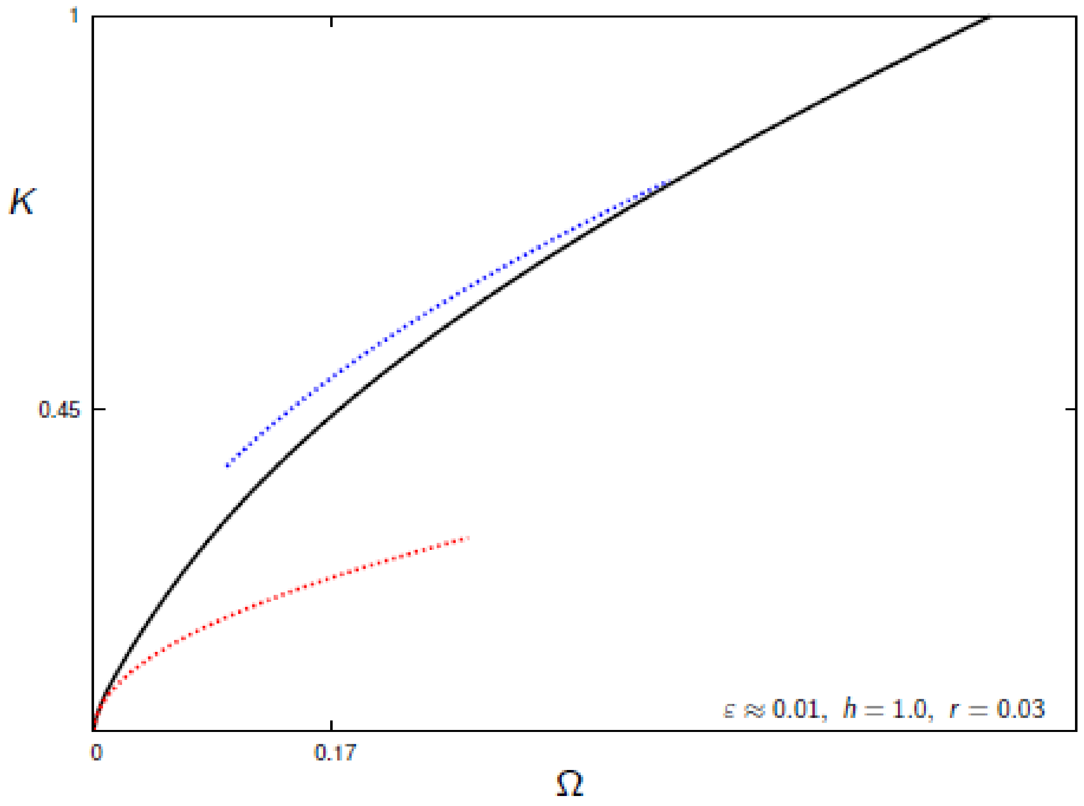


In the vicinity of zero frequency

$$G_1\Omega^2 + G_2K^4 = 0, \quad 0 < K \ll \sqrt{\varepsilon}, \quad \Omega \ll \varepsilon$$

At higher frequencies, including the vicinity of shear cut-off

$$G_3\Omega^2 + G_4K^4 = 0, \quad \sqrt{\varepsilon} \ll K \ll 1, \quad \varepsilon \ll \Omega \ll 1$$

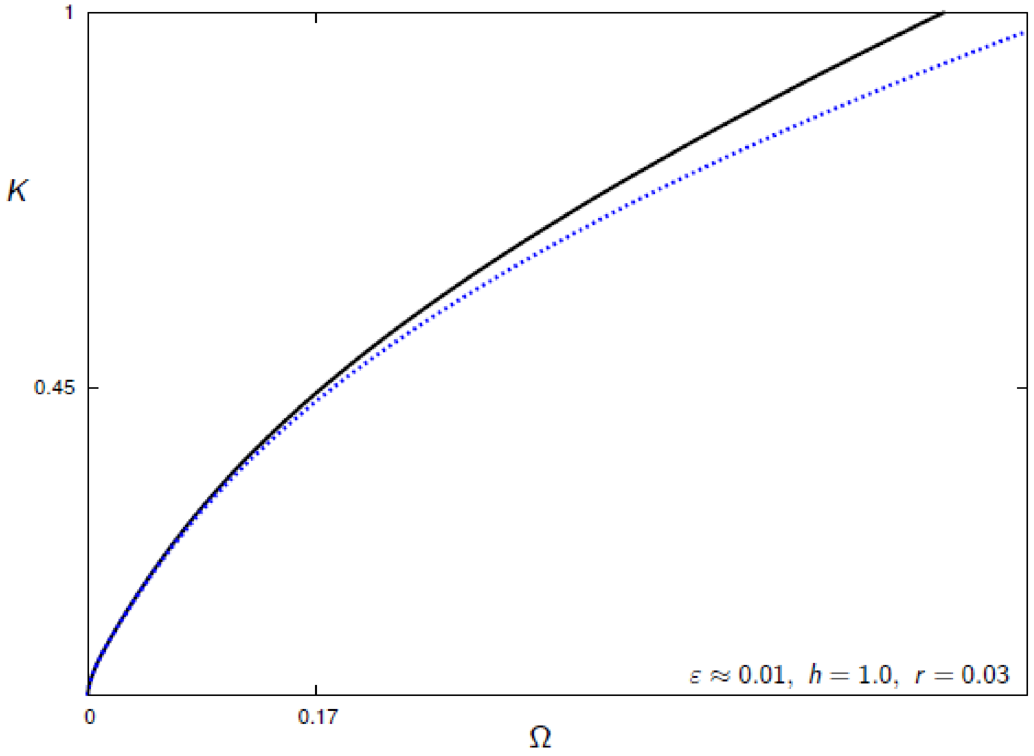


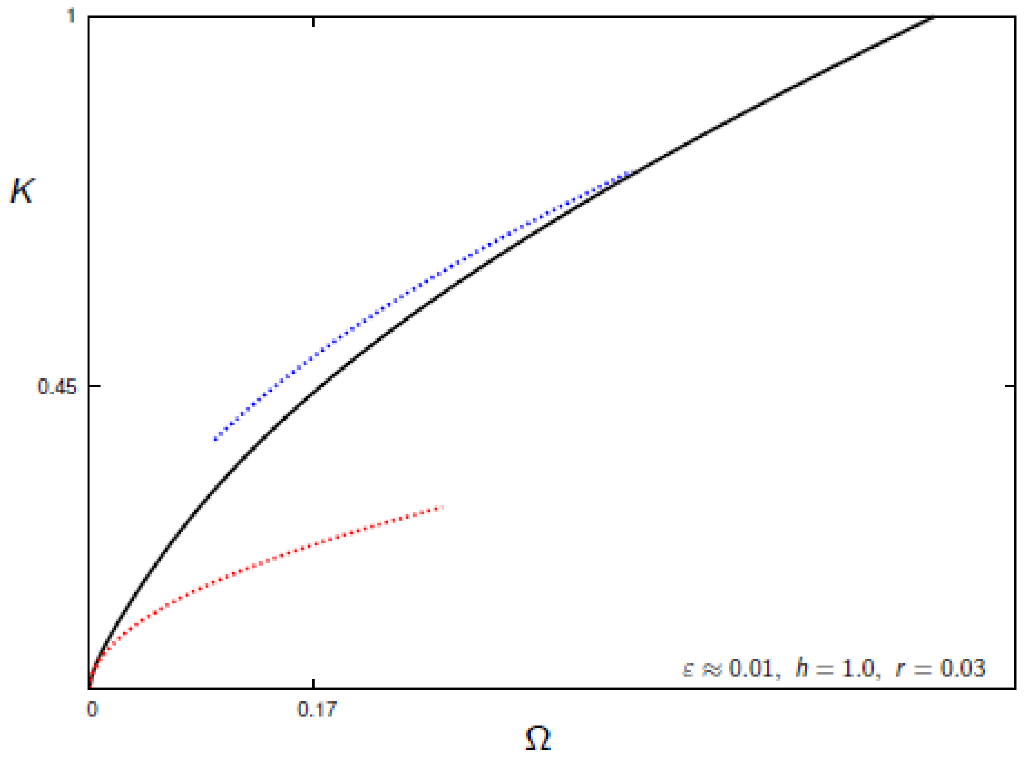
CLASSICAL KIRCHHOFF-TYPE THEORY IS NOT APPLICABLE!

Uniform approximation for the fundamental mode

Taking both local approximations we derive a uniform one:

$$G_1\varepsilon\Omega^2 + G_2\varepsilon K^4 + G_3 K^2\Omega^2 + G_4 K^6 = 0$$



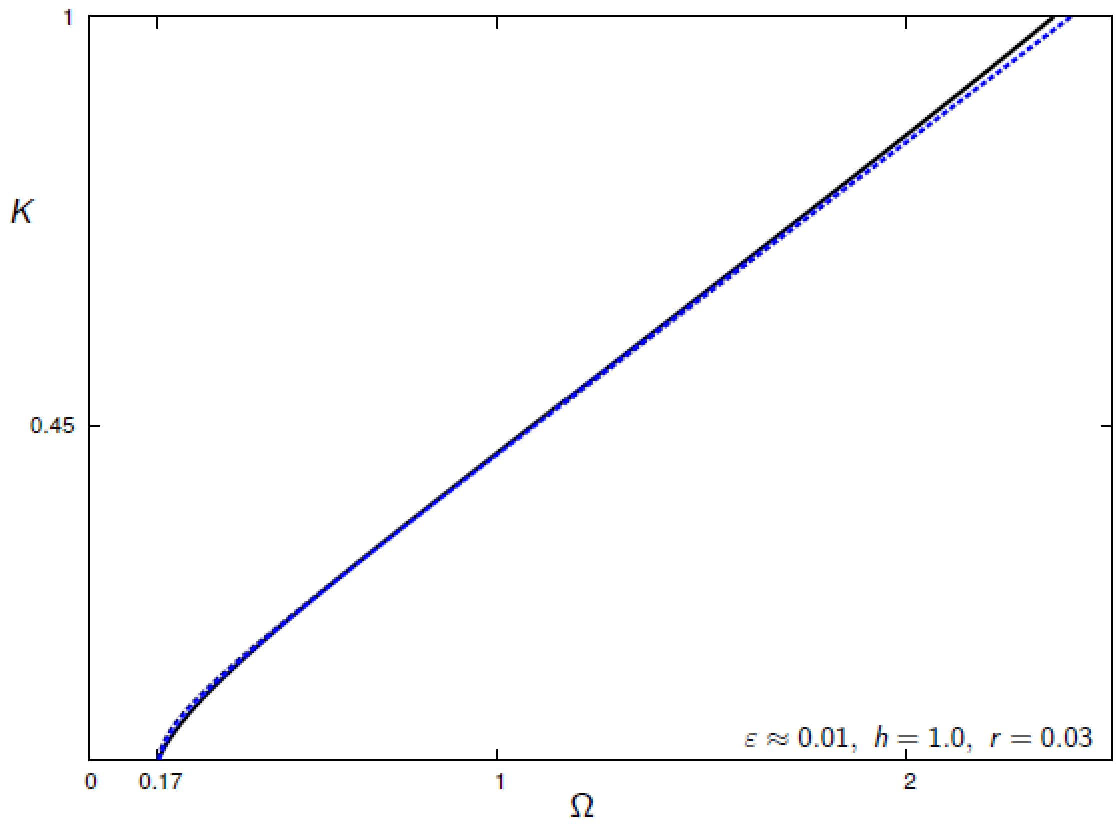


Also valid in the transition region $\Omega \sim \varepsilon, K \sim \sqrt{\varepsilon}$

Near shear cut-off approximation

For $\Omega \sim \sqrt{\varepsilon}$, $K \ll 1$

$$G_1\varepsilon + G_3K^2 + G_5\Omega^2 = 0$$



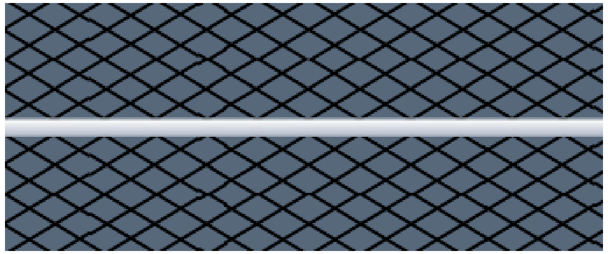
Low-frequency dispersion behaviour

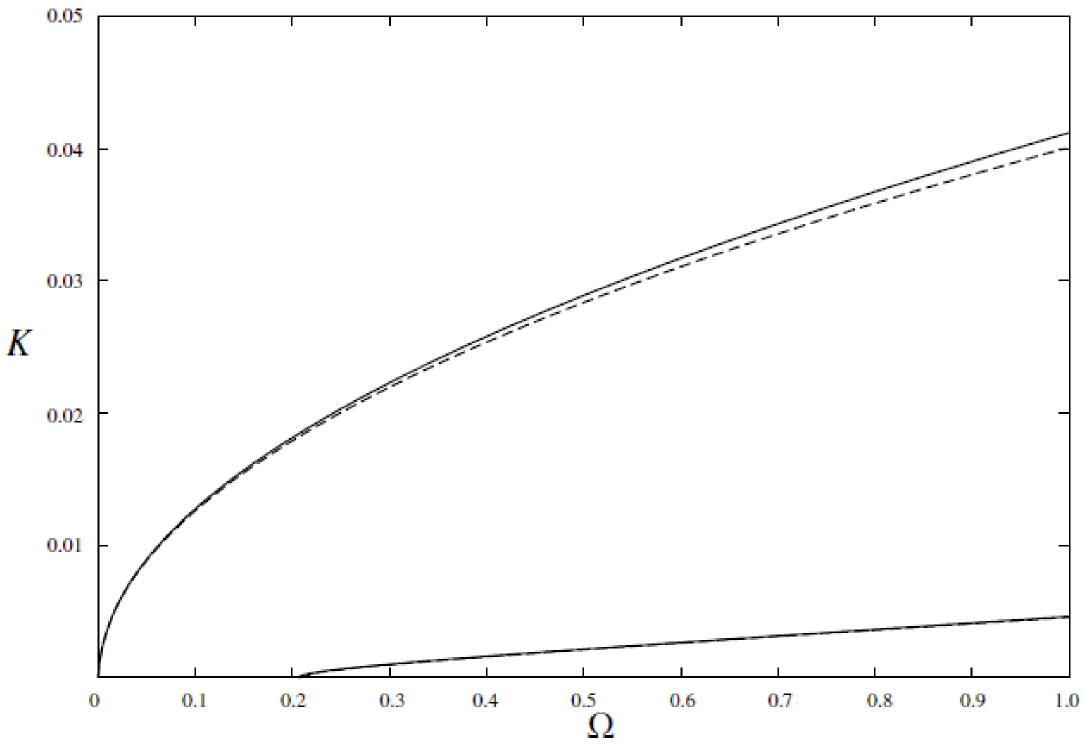
J. Kaplunov et al. *Proc. Eng.* 199 (2017): 1489-1494

B) Laminated glass

(stiff skin layers and light light core layer)

$$\varepsilon << 1, \; h \sim \varepsilon^{-1/4}, \; \rho \sim 1$$





UNIFORM TWO-MODE APPROXIMATIONS

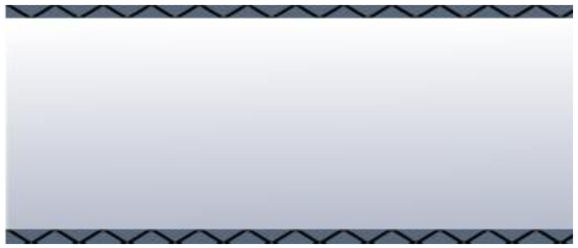
$$\varepsilon^{11/4} G_1 \Omega^2 + \varepsilon^{5/4} G_2 K^4 + \varepsilon^{3/2} G_3 K^2 \Omega^2 + G_4 K^6 + \varepsilon^{5/2} G_5 \Omega^4 = 0$$

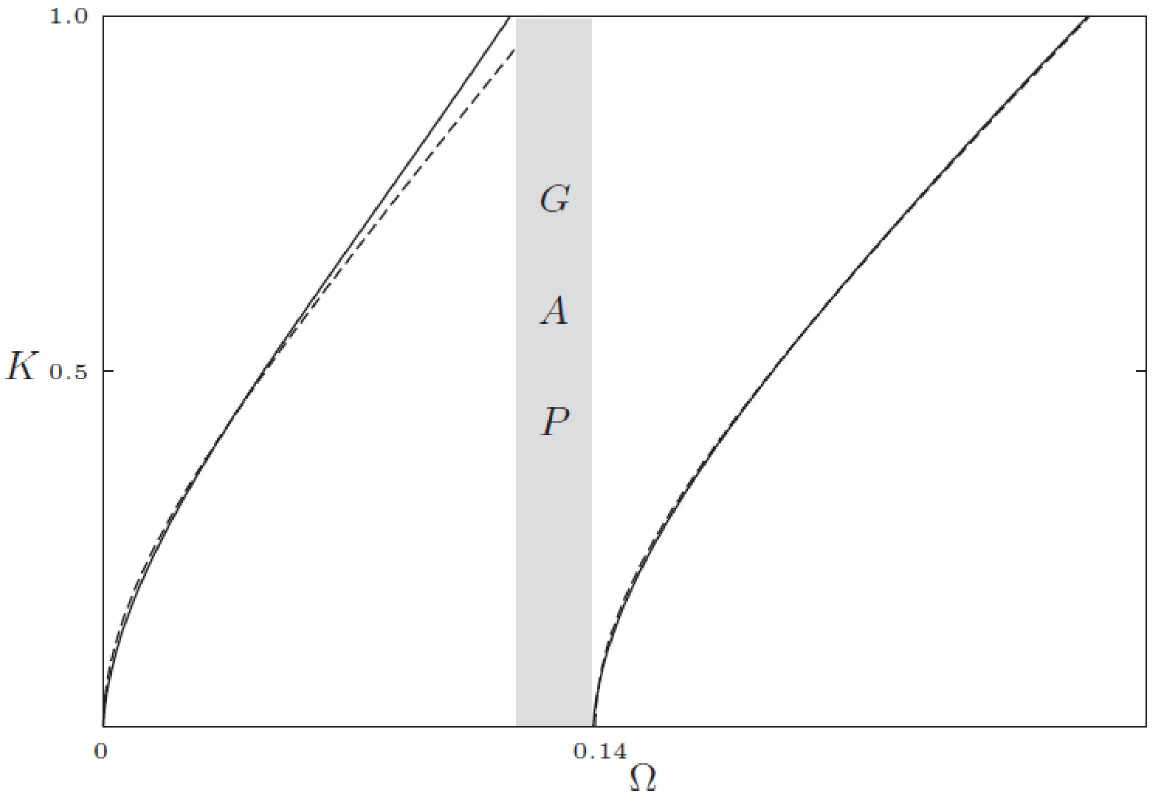
Low-frequency dispersion behaviour

C) Sandwich structure

$\varepsilon \ll 1, \quad h \sim \varepsilon, \quad r \sim \varepsilon^2$

(stiff thin skin layers and light core layer)

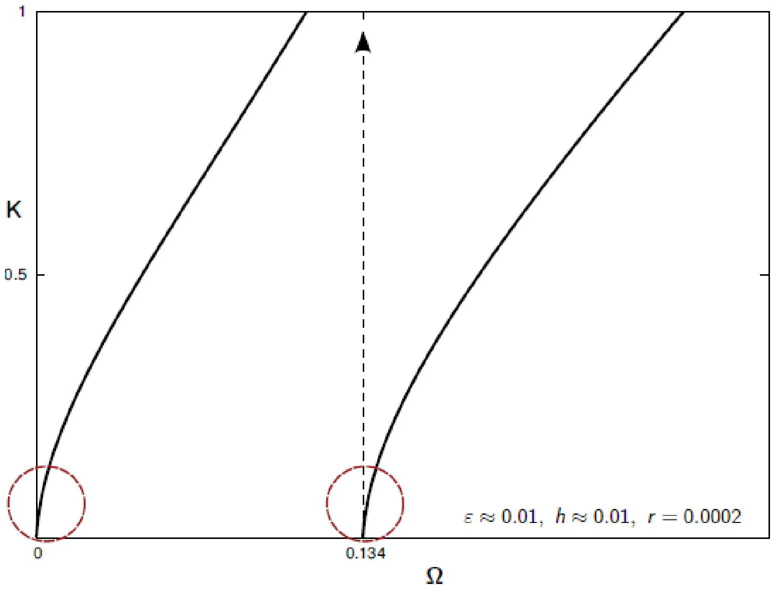


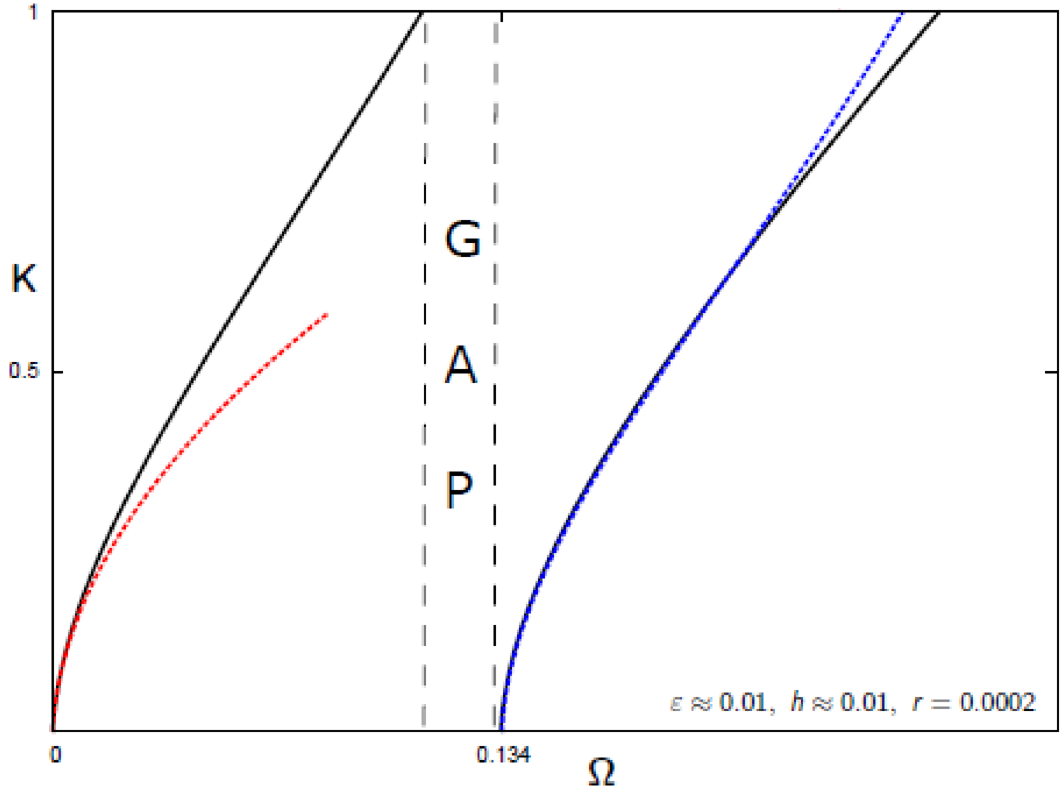


COMPOSITE APPROXIMATIONS

$$G_1 \varepsilon \Omega^2 + G_2 \varepsilon^2 K^4 + \varepsilon K^2 \Omega^2 \left(G_3 + \frac{r_0}{h_0} G_8 \right) + G_5 \Omega^4 = 0$$

Local approximations





NO OVERLAP REGION!

Fundamental mode

$$G_1\Omega^2 + G_2\varepsilon K^4 = 0,$$

$$K \ll 1, \quad \Omega \sim \sqrt{\varepsilon}K^2 \ll \sqrt{\varepsilon}$$

Shear mode

$$G_1\varepsilon + G_3\varepsilon K^2 + G_5\Omega^2 + G_8K^2\Omega^2 = 0,$$

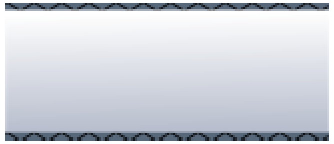
$$\varepsilon^{\frac{1}{2}} \ll K \ll 1, \quad \Omega \sim \sqrt{\varepsilon}$$

In progress

- Asymptotic models for strongly inhomogeneous plates

uniformly asymptotic

$$G_1 \varepsilon u_{tt} + G_2 \varepsilon \Delta^2 u + G_3 \Delta u_{tt} - G_4 \Delta^3 u + G_5 u_{tttt} = 0$$



composite

$$G_1 \varepsilon u_{tt} + G_2 \varepsilon^2 \Delta^2 u + G_5 u_{tttt} + \varepsilon \left(G_3 + \frac{r_0}{h_0} G_8 \right) \Delta u_{tt} = 0$$

- Boundary conditions
- Extension to layered shells

Concluding remarks

- ✓ Multi-component high-contrast elastic structures require specialised theory
- ✓ High contrast may lead to unexpectedly low natural frequencies
- ✓ Stronger components subject to Neumann conditions, perform almost rigid body motions
- ✓ Two-mode theories for long-wave low-frequency motion of layered plates can be asymptotically uniform or composite