Analytical Method for Space Debris propagation under perturbations in the geostationary ring

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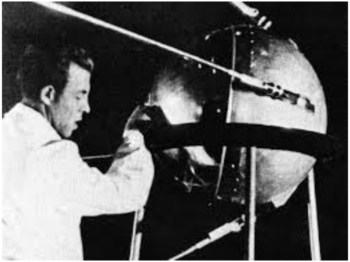
OUTLINE

- Introduction
- Motivation
- Methodology
- Results
- Applications
- Conclusion and Future work.

Artificial satellites

What was the first artificial satellite?

Sputnik I was the first artificial Earth satellite. It was 58 cm diameter polished metal sphere, with four external radio antennas to broadcast radio pulses. It was launched by the Soviet Union into an elliptical low Earth orbit on 4 October 1957.



Sputnik I

Current situation of satellites orbiting the Earth

Current data

According to NASA, the total number of launched satellites is **7463**. (July, 1^{st} 2016)
`http://nssdc.gsfc.nasa.gov/nmc/spacecraftSearch.do`

Discipline

The number of satellites (s/c) can be cataloged in different disciplines:

- Astronomy 319 s/c.
- Earth Science 946 s/c.
- Planetary Science 316 s/c.
- Solar and Space Physics 857 s/c.
- Human Crew 329 s/c.
- Life Science 97 s/c.
- Micro-gravity 72 s/c.
- **Communications** 2132 s/c.
- Engineering 419 s/c.
- Navigation and GPS 475 s/c.
- Resupply-Repair 215 s/c.
- **Surveillance and Military** 2299 s/c.
- Technology Applications 268 s/c.

Altitude classifications for geocentric orbits

Altitude classifications

Another way to classify the satellites is according to the altitude of the satellite with respect to the Earth surface.

- Low Earth Orbits **(LEO)**: altitudes up to $2,000$ km.
- Medium Earth Orbits **(MEO)**: altitudes from $2,000$ km. up to $35,786$ km.
- Geostationary Orbits **(GEO)**: altitudes of $35,786$ km. (ecc $= 0$, inc $= 0^{\circ}$)

- International Space Station (ISS) is in **LEO** region.
  The altitude is about 415 km.
  The velocity is 7.7 km/s.
- Other missions: Earth observation satellites, spy satellites...

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- Global Positioning System (GPS) is in **MEO** region. The altitude is about $20,200$ km. The velocity of the satellites is 3.8 km/s.
- Other missions: Navigation (GPS, Galileo), communication...

Altitude classifications for geocentric orbits

Altitude classifications

Another way to classify the satellites is according to the altitude of the satellite with respect to the Earth surface.

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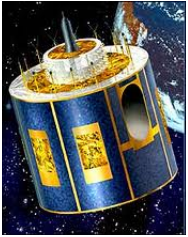
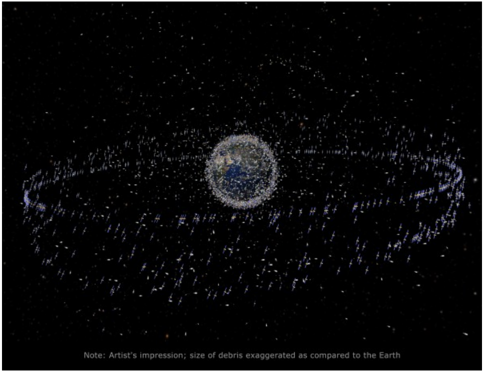


- Meteosat is in **GEO** region.
  The altitude is about $35,786$ km.
  The velocity is 3.07 km/s.
- Other missions: Weather forecast, meteorology, communications...

What is Space Debris?

Space Debris

Space debris are all man-made objects in orbit around the Earth which no longer serve a useful purpose. These objects are non-active satellites, fragments of satellites, rocket parts, remains of explosions or collisions, etc. of all sizes and all chemical compositions.

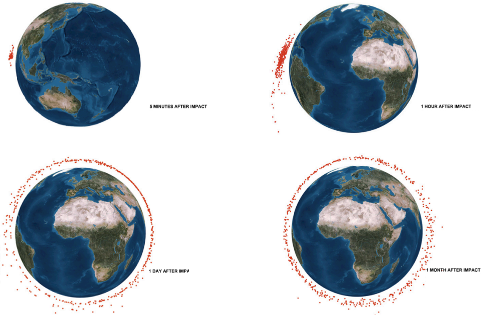


Space debris orbiting around the Earth. Three main congested regions: GEO, MEO, LEO.

Space Debris orbiting the Earth

The presented satellite missions can contribute to the huge collection of Space Debris orbiting the Earth through different ways:

- During the launch process.
- Intentional destruction of satellites. On January, 11^{th} 2007 China destroyed the weather satellite **Fengyun-1C** via an anti-satellite (ASAT) device.
-



Space Debris orbiting the Earth

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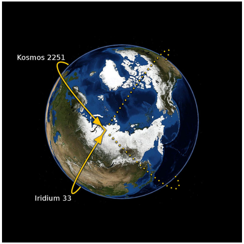
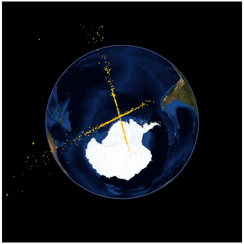
- During the launch process.
- Intentional destruction of satellites. On January, 11^{th} 2007 China destroyed the weather satellite **Fengyun-1C** via an anti-satellite (ASAT) device.
- Non-intentional explosions or collisions between satellites. The non-intentional collision between **Iridium 33** and **Kosmos-2251** on February, 10^{th} 2009.





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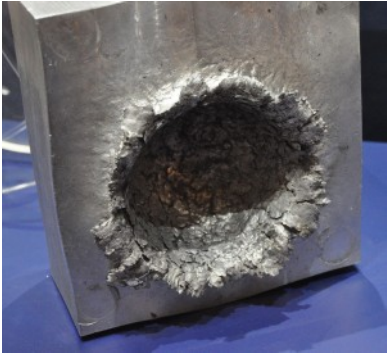
Space Debris orbiting the Earth

Space Debris travel at speeds up to **7.8 km/s**, fast enough to damage a satellite or a spacecraft.

A sample of what Space Debris can do

This 10.2 cm thick aluminum block was hit by a 2.5 cm, 15 gr plastic cylinder at 6.8 km/s. The plastic went almost all the way through the block, showing even plastic can damaged at orbital speeds and most space debris is metal, not plastic.

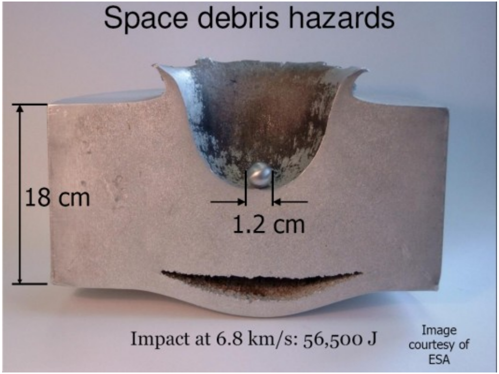


Space Debris orbiting the Earth

Space Debris travel at speeds up to **7.8 km/s**, fast enough to damage a satellite or a spacecraft.

A sample of what Space Debris can do

This 1.2 cm aluminum sphere striking a 18 cm thick aluminum plate at a velocity of 6.8 km/s, giving some idea of the destructive power of hyper-velocity impacts.



Space Debris orbiting the Earth

Space Debris travel at speeds up to 7.8 **km/s**, fast enough to damage a satellite or a spacecraft.

A sample of what Space Debris can do

A tiny piece of flying space debris hits the ISS last June, 12th.

Impacto en la cúpula de cristal de la Estación Espacial Internacional el pasado día 12. Es de 7 centímetros. /ESA

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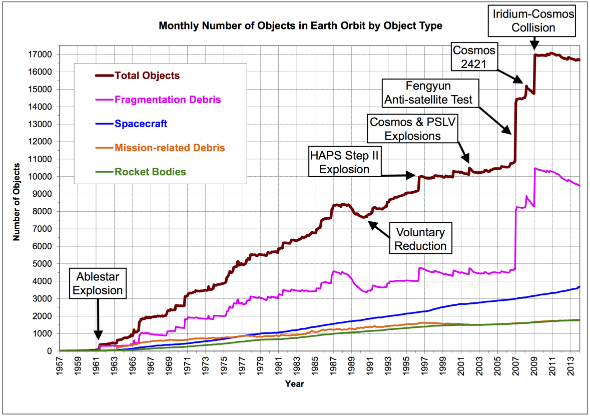
Current situation of space debris



- More than $21,000$ space debris larger than 10 cm.
- Approx. $500,000$ particles between 1 and 10 cm.
- More than 100 million pieces smaller than 1 cm.

Motivation

- ⋆ Where will space debris be located after 100 years?
- ⋆ Does the area-to-mass ratio of space debris affect its temporal evolution?
- ⋆ Will there be any stable region of space debris?

Methodology

Objective: Computation of the time-evolution of thousands of pieces of space debris to extract global properties, stable regions, etc.

Problem: The numerical approach is fast, but not enough to compute the evolution of thousands of pieces at the same time.

Solution: The analytical approach helps to strongly reduce the computational cost. Furthermore, it helps to study the periods of the eccentricity and inclination, to understand the influence of the area to mass ratio, etc.

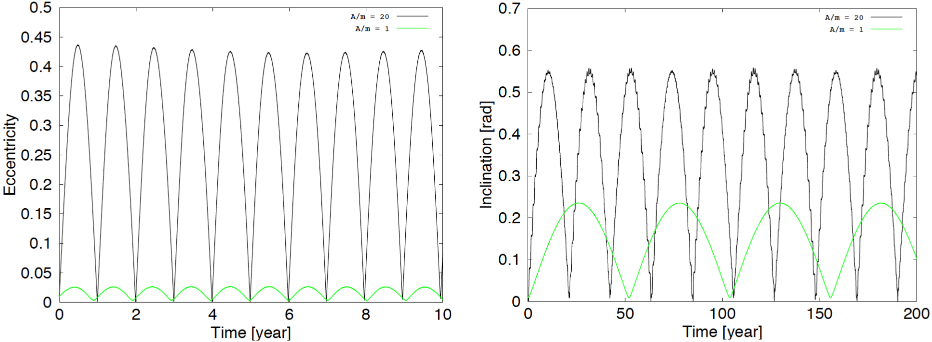
Tools :

- Hamiltonian formulation.
- Poincaré's variables.

Numerical simulations

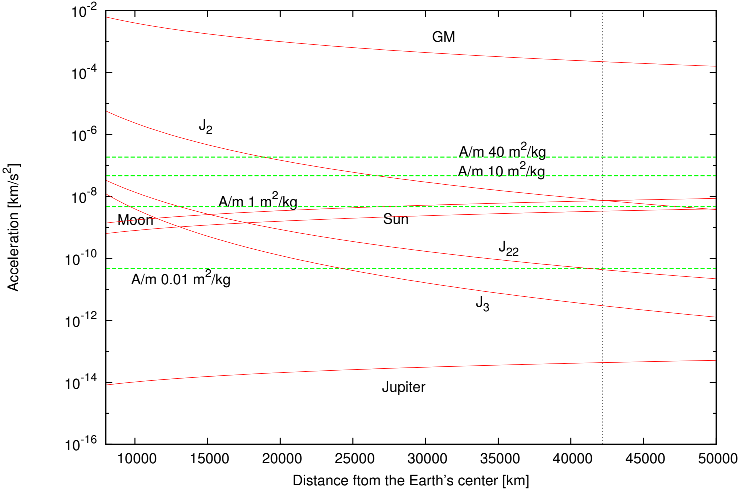
We consider an object located at the Geostationary ring under different perturbations: J_2 effect, Solar Radiation Pressure, Sun and Moon as third bodies.

- $a = 42.164$ km. $e = 0.01,$ $i = 0.01$ rad. $\omega = \Omega = M = 0$ rad.



Order of magnitude of perturbations

We consider an object located at the Geostationary ring under **different perturbations**: J_2 effect, Solar Radiation Pressure, Sun and Moon as third bodies.



State of the Art

Poincaré's variables

$$\begin{aligned} \mathbf{x_1} &= \sqrt{2P}\sin p, & \mathbf{y_1} &= \sqrt{2P}\cos p, \\ \mathbf{x_2} &= \sqrt{2Q}\sin q, & \mathbf{y_2} &= \sqrt{2Q}\cos q, \\ \lambda, & & \mathbf{L}, & \end{aligned}$$

where p, q, P, Q are the **modified Delaunay's elements** defined by:

$$\begin{aligned} P &= L - G, & p &= -\omega - \Omega, \\ Q &= G - H, & q &= -\Omega. \end{aligned}$$

where L, G, H are the **classical Delaunay's elements**:

$$L = \sqrt{\mu a}, \qquad G = \sqrt{\mu a(1-e^2)}, \qquad H = \sqrt{\mu a(1-e^2)}\cos i.$$

Finally, λ is the mean longitude $\lambda = M + \omega + \Omega$.

Classical Orbital Elements

Poincaré's variables are:

- especially useful for treating **problems with Hamiltonian dynamics**.
- suitable for **all eccentricities and inclinations**.

Space debris as a Dynamical System

Hamiltonian formulation of the problem

Given the generalized coordinates $\mathbf{q} = (x_1, x_2, \lambda)$ and the generalized momenta $\mathbf{p} = (y_1, y_2, L)$ of a piece of space debris orbiting around the Earth, it is possible to describe its motion following the **Hamiltonian formulation**:

$$\begin{cases} \dot{q}_j = \dfrac{\partial H}{\partial p_j}, \\ \dot{p}_j = -\dfrac{\partial H}{\partial q_j}, \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \dfrac{\partial H}{\partial y_1}, & \dot{y}_1 = -\dfrac{\partial H}{\partial x_1}, \\ \dot{x}_2 = \dfrac{\partial H}{\partial y_2}, & \dot{y}_2 = -\dfrac{\partial H}{\partial x_2}, \\ \dot{\lambda} = \dfrac{\partial H}{\partial L}, & \dot{L} = -\dfrac{\partial H}{\partial \lambda}. \end{cases}$$

The Hamiltonian expression in terms of Poincaré's variables is:

$$H(\mathbf{q},\mathbf{p}) = H_{kepler}(\mathbf{q},\mathbf{p}) + H_{SRP}(\mathbf{q},\mathbf{p}) + H_{J_2}(\mathbf{q},\mathbf{p}) + H_{3bS}(\mathbf{q},\mathbf{p}) + H_{3bM}(\mathbf{q},\mathbf{p}).$$

Hamiltonian formulation of the problem

State of the art

The Hamiltonian considered in a previous model[a] was:

$$H = H_{kepler} + H_{SRP}.$$

[a]Hubaux, C., Lemaître, A.: "The impact of Earth's shadow on the long-term evolution of space debris," *Celest. Mech. Dyn. Astr.,* Vol. 116, 1, 79-95 (2013)

However, the present research includes the J_2 effect in the Hamiltonian formulation and the third body effect due to the Sun and Moon:

$$H(\mathbf{q},\mathbf{p}) = \underbrace{H_{kepler}(\mathbf{q},\mathbf{p})}_{H_{Kepler}} + \underbrace{H_{SRP}(\mathbf{q},\mathbf{p})}_{H_{SRP}} + \underbrace{H_{J_2}(\mathbf{q},\mathbf{p})}_{H_{J_2}} + \underbrace{H_{3bS}(\mathbf{q},\mathbf{p})}_{H_{3bS}} + \underbrace{H_{3bM}(\mathbf{q},\mathbf{p})}_{H_{3bM}}.$$

The main reasons for including these perturbations are:

- The acceleration caused by SRP is stronger than J_2 or vice-versa depending on the area-to-mass ratio for space debris located at the **geostationary ring**.
- The third body perturbation (Sun and Moon) must be considered, especially for long-term propagation. Order of Perturbations

First averaging process: Space debris (1-day-period)

The Hamiltonian is averaged over the **mean longitude (λ)** since, for long-time propagations, the short periodic oscillations caused by λ are meaningless.

Direct consecuences

- 6 d.o.f $(x_1, y_1, x_2, y_2, \lambda, L)$ to 4 d.o.f (x_1, y_1, x_2, y_2).
- λ is not present anymore. Semi-major axis becomes constant.

$$\overline{H}(\mathbf{q},\mathbf{p}) = \underbrace{\overline{H}_{kepler}(\mathbf{q},\mathbf{p})}_{\overline{H}_{Kepler}} + \underbrace{\overline{H}_{SRP}(\mathbf{q},\mathbf{p})}_{\overline{H}_{SRP}} + \underbrace{\overline{H}_{J_2}(\mathbf{q},\mathbf{p})}_{\overline{H}_{J_2}} + \underbrace{\overline{H}_{3bS}(\mathbf{q},\mathbf{p})}_{\overline{H}_{3bS}} + \underbrace{\overline{H}_{3bM}(\mathbf{q},\mathbf{p})}_{\overline{H}_{3bM}}$$

The following dynamical system provides the time evolution of the Poincaré's variables (limiting to SRP_1 and J_2):

$$\begin{cases} \dot{x}_1(t) = \dfrac{\partial \overline{\mathcal{H}}}{\partial y_1}, \quad \dot{x}_2(t) = \dfrac{\partial \overline{\mathcal{H}}}{\partial y_2}, \\ \dot{y}_1(t) = -\dfrac{\partial \overline{\mathcal{H}}}{\partial x_1}, \quad \dot{y}_2(t) = -\dfrac{\partial \overline{\mathcal{H}}}{\partial x_2}. \end{cases}$$

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The explicit solution is given by (with $\eta = \frac{C_2}{n_\odot}$):

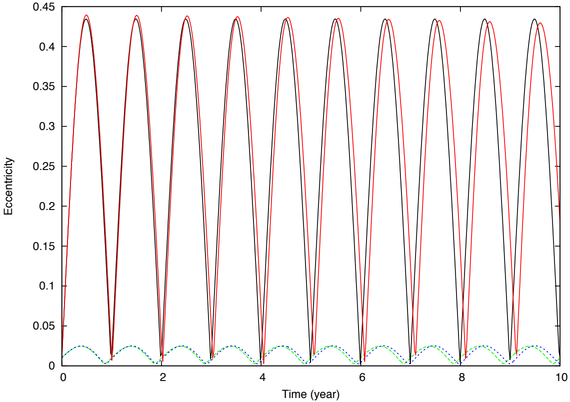
$$x_1(t) = \mathcal{A}\,\sin(C_2 t + \Phi) + \frac{k\sin(n_\odot t + \lambda_{\odot,0})}{1-\eta^2}\left[\eta\,\cos\epsilon + 1\right],$$

$$y_1(t) = \mathcal{A}\,\cos(C_2 t + \Phi) + \frac{k\cos(n_\odot t + \lambda_{\odot,0})}{1-\eta^2}\left[\cos\epsilon + \eta\right],$$

where the constants $\mathcal{A}$ and Φ are determined by the initial conditions.

Evolution of the eccentricity

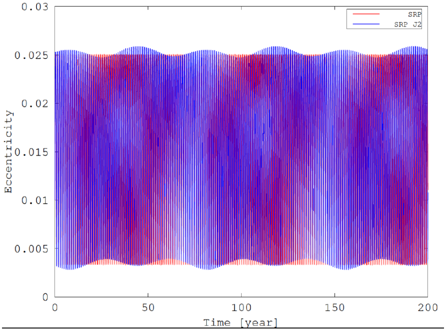
The 1-year periodic motion is the most important. Its amplitude is proportional to k, which means proportional to the area-to-mass ratio.



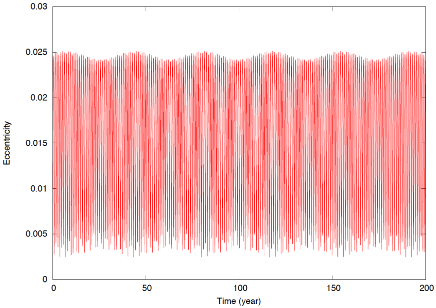
- Eccentricity evolution during 10 years with $A/m = 1, A/m = 20$.
- Considered perturbations: SRP with/without the J_2 effect.

Evolution of the eccentricity

Motion of the eccentricity over 200 years and comparison with a similar numerical integration; we clearly see the superposition of the long and short motions on both figures.



(a) Numerical.



(b) Analytical.

Second averaging process: Sun (1-year-period) and Moon (1-month-period)

We have averaged the equations $\dot{x_2}(t)$ and $\dot{y_2}(t)$ over the variables $\lambda_\odot$ and $\lambda_{☾}$, and we obtain the corresponding solution for $x_2(t)$ and $y_2(t)$:

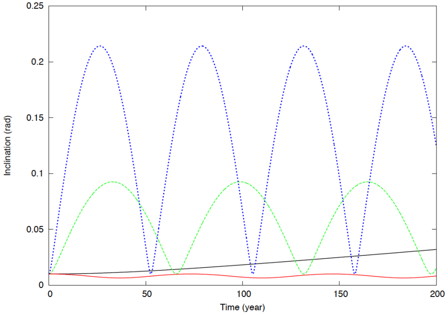
$$\boxed{\begin{aligned} x_2(t) &= \mathcal{D}\ \sin(\sqrt{d_1 d_2}\, t - \psi), \\ y_2(t) &= \mathcal{D}\ \sqrt{\frac{d_2}{d_1}}\ \cos(\sqrt{d_1 d_2}\, t - \psi) - \frac{d_3}{d_1}. \end{aligned}}$$

Expressions for d_1, d_2 y d_3

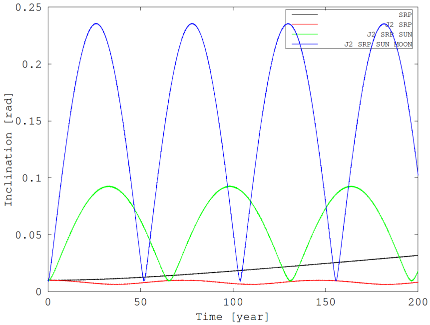
Remarks

- These equations represent an oscillatory motion; $\mathcal{D}$ is the amplitude and ψ the phase space.
- x_1, y_1, x_2 and y_2 represent the analytical solution of the problem of space debris orbiting around the Earth in the geostationary ring.

Evolution of the inclination : 200 years with $A/m = 1$

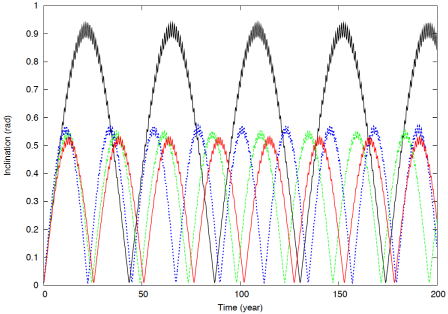


(c) Our analytical model.

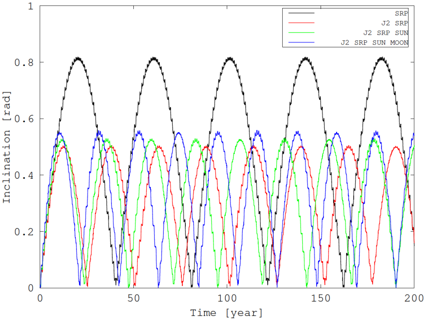


(d) Numerical integration NIMASTEP.

Evolution of the inclination : 200 years with $A/m = 20$



(a) Our analytical model

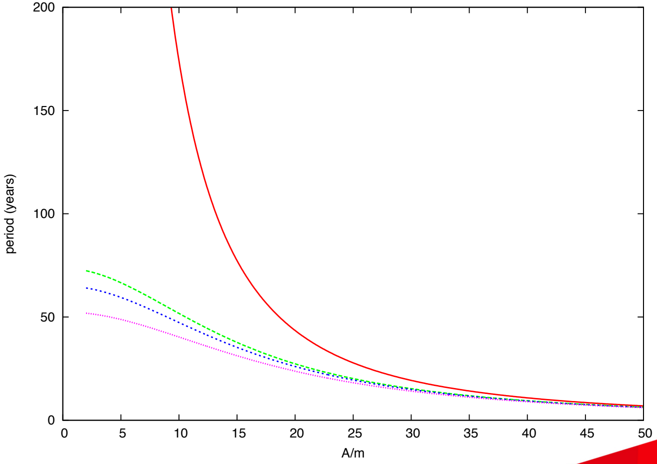


(b) Numerical integration NIMASTEP

Period of the inclination as a function of A/m (m^2/kg)

Each perturbation reduces the period of the inclination.

- SRP (red).
- $SRP + J_2$ (green).
- $SRP + J_2 +$ Sun (blue).
- $SRP + J_2 +$ Sun $+$ Moon (Magenta).



Application : Synthetic population of space debris

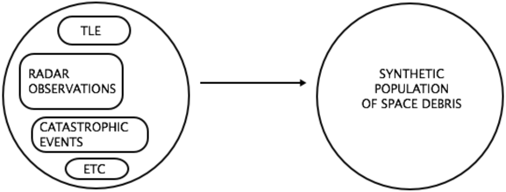
Synthetic population in the GEO region and the analytical approach

Real population Unknown number of pieces of space debris (21.000 objects greater than 10 cm according to NASA.)

Goal Simulate and understand evolution of pieces of space debris orbiting around the Earth, based on the characteristics of each individual (initial orbital elements, ratio A/m, etc)

Problem It is not possible to analyze all the individuals.

Solution Construct an **artificial (synthetic) population** starting from known data around the true one (Two Line Elements, radar observations, etc) and approximating the structure of the true population as accurate as possible by means of the **analytical method**.



Conclusions and Future Work

Conclusions

- An analytical model is presented to propagate space debris in the GEO region.
- A study of the period of the inclination is presented and the influence on it of the different perturbations.
- A numerical study has been performed to compare the analytical solution with the numerical one.

Future work

- **Design a powerful synthetic population** that will include all the catalogued space debris plus thousands of artificial debris in the GEO region.
- Extract **global properties or predictions** of the synthetic population.
- Extract **properties** of the observed objects in the GEO region. Observation Space Debris

Thanks for your attention!

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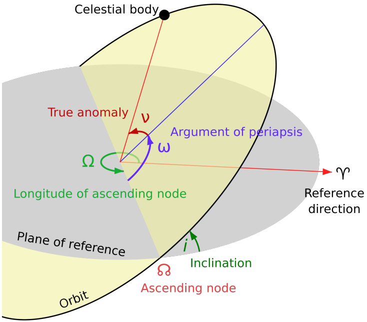
Classical Orbital Elements

The **Classical Orbital Elements** are the parameters required to uniquely identify a specific orbit. The traditional orbital elements are the six Keplerian elements.

- Semi-major axis (a).
- Eccentricity (e).
- Inclination (i).
- Argument of perigee (ω).
- Longitude of the ascending node (Ω).
- Mean anomaly (M).



Poincaré's variables

Hamiltonian formulation: H_{kepler}

H_{kepler} represents the attraction of the Earth as a central body. Its formulation is:

$$H_{kepler}(\mathbf{r},\mathbf{v}) = \frac{v^2}{2} - \frac{\mu}{r},$$

where $\mu = \mathcal{G}M_{\oplus}$ with $\mathcal{G}$ the standard gravitational constant, $M_{\oplus}$ the mass of the Earth.

State of the art

Hamiltonian formulation: H_{SRP}

H_{SRP} represents the direct solar radiation pressure potential. Its expression is:

$$\begin{aligned} H_{SRP}(\mathbf{r},\mathbf{r}_\odot) &= C_r\, P_r\, \frac{A}{m} a_\odot^2 \frac{1}{||\mathbf{r}-\mathbf{r}_\odot||} \\ &= C_r P_r \frac{A}{m} a_\odot \sum_{n\geq 0} \left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi), \end{aligned}$$

- C_r (fixed to 1 in this work) is a dimension-free reflectivity coefficient.
- $Pr = 4.56 \times 10^{-6}\ N/m^2$ is the radiation pressure for an object located at a distance of 1 AU from the Sun.
- A/m is the area-to-mass ratio.
- $a_\odot$ is equal to the mean distance between the Sun and the Earth (i.e. $a_\odot = 1$ AU).
- $r_\odot \simeq a_\odot$.

We split the expression in three parts:

$$\begin{aligned} H_{SRP}(\mathbf{r},\mathbf{r}_\odot) &= C_r P_r \frac{A}{m} a_\odot (1 + \frac{r}{a_\odot}\cos(\phi)) + C_r P_r \frac{A}{m} a_\odot \sum_{n\geq 2} \left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi) \\ &\simeq H_{SRP_0} + H_{SRP_1} + H_{SRP_2}, \end{aligned}$$

State of the art

Hamiltonian formulation: H_{J_2}

H_{J_2} represents the potential function that affects space debris due to the Earth oblateness. In this work we only consider the zonal harmonic J_2, which is the most representative of the potential function.

The expression of H_{J_2} in terms of the position is:

$$\begin{aligned} H_{J_2}(\mathbf{r}) &= \frac{\mu}{r} J_2 \left(\frac{r_\oplus}{r}\right)^2 P_2\left(\sin\phi_{sat}\right) \\ &= \frac{\mu}{r} J_2 \left(\frac{r_\oplus}{r}\right)^2 \frac{1}{2}\left(3\left(\frac{z}{r}\right)^2 - 1\right), \end{aligned}$$

where ϕ_{sat} represents the latitude of the satellite, and consequently, $\sin\phi_{sat} = z/r$.

State of the art

Hamiltonian formulation: H_{3bS} and H_{3bM}

The **solar perturbation** can be expressed by:

$$
\begin{aligned}
H_{3bS}(\mathbf{r},\mathbf{r}_\odot) &= -\mu_\odot \frac{1}{||\mathbf{r}-\mathbf{r}_\odot||} + \mu_\odot \frac{\mathbf{r}\cdot\mathbf{r}_\odot}{||\mathbf{r}_\odot||^3} \\
&= -\frac{\mu_\odot}{a_\odot}\sum_{n\geq 0}\left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi) + \mu_\odot \frac{r a_\odot \cos(\phi)}{a_\odot^3} \\
&= -\frac{\mu_\odot}{a_\odot}(1+\sum_{n\geq 2}\left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi)),
\end{aligned}
$$

where $\mu_\odot = \mathcal{G}M_\odot$ with $M_\odot$ the mass of the Sun.

Similarly, the **lunar perturbation** writes:

$$
H_{3bM}(\mathbf{r},\mathbf{r}_☾) = -\frac{\mu_☾}{a_☾}(1+\sum_{n\geq 2}\left(\frac{r}{a_☾}\right)^n P_n(\cos\phi_M)),
$$

where $\mu_☾ = \mathcal{G}M_☾$ with $M_☾$ the mass of the Moon, and ϕ_M representing the angle between the satellite and the Moon positions. The Moon is also assumed to follow a circular orbit, i.e. $r_☾ = a_☾$.

State of the art

Averaged Hamiltonian: $\overline{H}_{kepler}$

$\overline{H}_{kepler}$ represents the averaged Hamiltonian of the attraction of the Earth as a central body,

$$\overline{H}_{kepler}(\mathbf{q}, \mathbf{p}) = -\frac{\mu^2}{2L^2}.$$

It is now a constant term and will be omitted.

▸ Averaged Hamiltonian

Averaged Hamiltonian: $\overline{H}_{J_2}$

$\overline{H}_{J_2}(\mathbf{q}, \mathbf{p})$ is obtained using the averaged Lagrange Planetary Equations:

$$\begin{aligned} \frac{d\omega}{dt} &= \frac{3}{4}\sqrt{\frac{\mu}{a^3}} J_2 \frac{r_\oplus^2}{a^2} \frac{4-5\sin^2 i}{(1-e^2)^2} = \frac{C_2}{2} \frac{4-5\sin^2 i}{(1-e^2)^2}, \\ \frac{d\Omega}{dt} &= -\frac{3}{2}\sqrt{\frac{\mu}{a^3}} J_2 \frac{r_\oplus^2}{a^2} \frac{\cos i}{(1-e^2)^2} = -C_2 \frac{\cos i}{(1-e^2)^2}, \end{aligned} \tag{1}$$

where $C_2 = \frac{3}{2}\sqrt{\frac{\mu}{a^3}} J_2 \frac{r_\oplus^2}{a^2}$.

Following the Hamiltonian formulation:

$$\begin{aligned} \dot{p} &= -\dot{\omega} - \dot{\Omega} = \frac{\partial H_{J_2}}{\partial P} = C_p, \\ \dot{q} &= -\dot{\Omega} = \frac{\partial H_{J_2}}{\partial Q} = C_q, \end{aligned}$$

and consequently, if we choose constant values for C_p and C_q,

$$\overline{H}_{J_2}(x_1, y_1, x_2, y_2) = C_p\, P + C_q\, Q = \frac{C_p}{2}(x_1^2 + y_1^2) + \frac{C_q}{2}(x_2^2 + y_2^2).$$

In the case of $e = 0$ and $i = 0$, we obtain : $C_p = -C_2$ and $C_q = C_2$.

Averaged Hamiltonian

Averaged Hamiltonian: $\overline{H}_{SRP}$

The Hamiltonian expression for the direct solar radiation pressure can be split into:

$$\overline{H}_{SRP}(x_1, y_1, x_2, y_2) = H_{SRP_0} + H_{SRP_1} + H_{SRP_2},$$

- H_{SRP_0} is a constant term and will be omitted.
- Following (Hubaux and Lemaitre 2013) $\overline{H}_{SRP_1}$ can be expressed in terms of Poincaré's variables, truncated at e^2 or i^2, as:

$$H_{SRP_1} = -n_{\odot} k \left[r_{\odot,1}(x_1 R_2 + y_1 R_1) - r_{\odot,2}(x_1 R_3 + y_1 R_2) - r_{\odot,3}(x_1 R_5 - y_1 R_4) \right].$$

- H_{SRP_2} = ?

The second order part of the solar radiation pressure, the solar and lunar perturbations have also to be averaged over the fast variable λ.

We obtain an expression for:

- $\overline{H}_{SRP_2} + \overline{H}_{3bS} = \overline{H}_{SRP_2+3bS}$ = ?
- $\overline{H}_{3bM}$ = ?

Averaged Hamiltonian

Averaged Hamiltonian: $\overline{H}_{SRP_2+3bS}$

These are second order terms in $\frac{a}{a_\odot}$, then we limit their expansion to the first term, neglecting the following terms proportional to e^2. In other words, we keep the terms in $\frac{a\,e^2}{a_\odot}$ but not those in $(\frac{a\,e}{a_\odot})^2$. The immediate consequence is the dependence of the averaged perturbation $\overline{H}_{SRP_2+3bS}$ only on x_2 and y_2 and not on x_1 and y_1:

$$\overline{H}_{SRP_2+3bS} = \overline{H}_{SRP_2+3bS}(-,-,x_2,y_2,\mathbf{r}_\odot) + O(\frac{a^2e^2}{a_\odot^2}).$$

With this assumption, we obtain :

$$\overline{H}_{SRP_2+3bS}(x_2,y_2,\mathbf{r}_\odot) = -\left[C_r P_r \frac{A}{m} a_\odot - \frac{\mu_\odot}{a_\odot}\right] \frac{3a^2}{4a_\odot^2} \upsilon_S^2 = -\beta\,\frac{3a^2}{4a_\odot^2} \upsilon_S^2,$$

where $\upsilon_S = \upsilon_S(x_2,y_2,\mathbf{r}_\odot) = -\sin q\ \sin i\ \mathbf{r}_{\odot,1} - \cos q\ \sin i\ \mathbf{r}_{\odot,2} + \cos i\ \mathbf{r}_{\odot,3}$, and

$$\beta = \left[C_r P_r \frac{A}{m} a_\odot - \frac{\mu_\odot}{a_\odot}\right].$$

▸ Averaged Hamiltonian

Averaged Hamiltonian: $\overline{H}_{3bM}$

With the same assumptions, the lunar perturbation is given by:

$$\overline{H}_{3bM}(x_2, y_2, \mathbf{r}_{☾}) = \frac{\mu_{☾}}{a_{☾}} \frac{3a^2}{4a_{☾}^2} \upsilon_M^2,$$

where $\upsilon_M = -\sin q \, \sin i \, \mathbf{r}_{☾,1} - \cos q \, \sin i \, \mathbf{r}_{☾,2} + \cos i \, \mathbf{r}_{☾,3}$.

Averaged Hamiltonian

Second averaging process: Sun (1-year-period) and Moon (1-month-period)

We have averaged the equations $\dot{x_2}(t)$ and $\dot{y_2}(t)$ over the variables $\lambda_\odot$ and $\lambda_☾$, and we obtain the following simplified linear equations:

$$\begin{cases} \dot{x_2}(t) = & d_1\ y_2 + d_3, \\ \dot{y_2}(t) = & -d_2\ x_2, \end{cases}$$

where

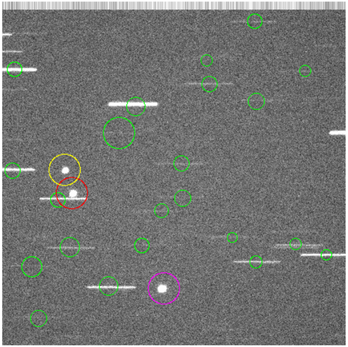
$$\begin{aligned} d_1 &= n_\odot \frac{k^2}{4L}\ \cos\epsilon + \frac{C_q}{2} - \delta - \delta\ \cos 2\epsilon - \gamma - \gamma\ \cos\epsilon_M, \\ d_2 &= n_\odot \frac{k^2}{4L}\ \cos\epsilon + \frac{C_q}{2} - 2\,\delta\ \cos 2\epsilon - 2\,\gamma\ \cos 2\epsilon_M, \\ d_3 &= -n_\odot \frac{k^2}{2\sqrt{L}}\ \sin\epsilon + 2\,\delta\,\sqrt{L}\ \sin 2\epsilon + 2\,\gamma\,\sqrt{L}\ \sin 2\epsilon_M, \end{aligned}$$

where $\delta = \beta\,\frac{3a^2}{16\ L\ a_\odot^2}$ and $\gamma = -\frac{\mu_☾}{a_☾}\ \frac{3a^2}{16\ L\ a_☾^2}$.

Second Averaging Process

Procedure

- Observation and detection of space debris in the GEO region.
- Orbit determination $(a, e, i, \omega, \Omega, M)$.
- Once we know the inclination evolution of the object, we will be able to adjust the area-to-mass ratio of the object by fitting the analytical approach. Conclusions



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