The Minkowski-Lorentz space and the spheres space: a survey

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Abstract

This document deals with the Computer Aided Geometric Design with a short presentation of the Minkowski-Lorentz space. This space generalizes to $\mathbb{R}^5$ the one used in the relativity theory. The Minkowski-Lorentz space offers a more intuitive writing of a sphere given by a point, a normal vector at the point and its curvature. It also eases the use of canal surfaces thus represented by curves. The quadratic computation in $\mathbb{R}^3$ becomes linear in that space. The use of spheres, canal surfaces and their particular case known as Dupin cyclides is illustrated in a schematic seahorse. The seahorse applies the G^1 connection in the Minkowski-Lorentz space.

Oriented spheres and Pencils

An oriented sphere S with centre Ω and radius $r > 0$ satisfies the relationship $\overrightarrow{\Omega M} = \rho \overrightarrow{N}$ with the rule $\rho = r$ (resp. $\rho = -r$) if the unit normal vector $\overrightarrow{N}$ to the sphere at point M is getting outside (resp. inside). The power of the point M to the sphere S is defined by $\chi_S(M) = \Omega M^2 - \rho^2$. The set of points solution of $\lambda_1 \chi_{S_1}(M) + \lambda_2 \chi_{S_2}(M) = 0$ is called the spheres pencil defined by S_1 et S_2.
There kinds of pencils exist, a circle based pencil, a tangent spheres pencil, a limited points pencil.

The Minkowski-Lorentz space

The quadratic form of Lorentz is defined on the basis $(\overrightarrow{e_o}; \overrightarrow{e_1}; \overrightarrow{e_2}; \overrightarrow{e_3}; \overrightarrow{e_\infty})$ by $\mathcal{Q}_{4,1}(x_o, x, y, z, x_\infty) = x^2 + y^2 + z^2 - 2\,x_o\,x_\infty$. The light cone C_l satisfies the equation $x^2 + y^2 + z^2 - 2\,x_o\,x_\infty = 0$ in the frame $(O_5; \overrightarrow{e_o}; \overrightarrow{e_1}; \overrightarrow{e_2}; \overrightarrow{e_3}; \overrightarrow{e_\infty})$. The unit sphere Λ^4 with centre O_5 in $\mathbb{R}^5$ is given by :

$$\Lambda^4 = \left\{ \sigma \in \mathbb{R}^5 \,|\, \mathcal{Q}_{4,1}\left(\overrightarrow{O_5\sigma}\right) = \overrightarrow{O_5\sigma}^2 = 1 \right\}$$

It represents the oriented spheres and planes of $\mathbb{R}^3$. A sphere or a plane S is represented by a point σ of $\mathbb{R}^5$.

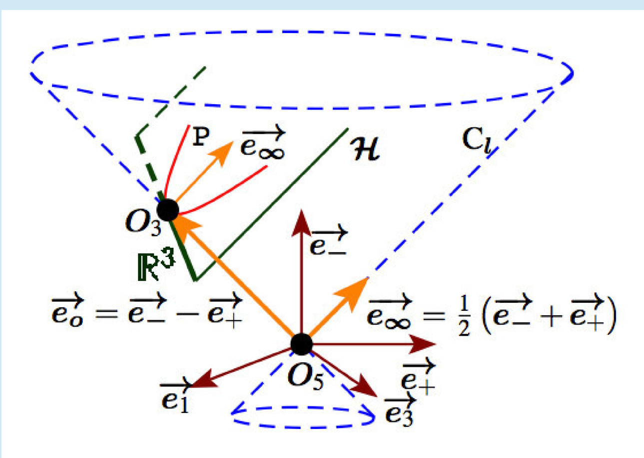

Figure 1: The Minkowski-Lorentz space

Linear pencils of spheres on Λ^4

On the unit sphere Λ^4 any pencil of sphere is represented by the intersection $\mathcal{C} = \Lambda^4 \cap \mathcal{P}$ of a plane called 2-plane $\mathcal{P}$ passing through O_5. $\mathcal{C}$ is a unit circle seen differently depending on the type of plane.

- If $\mathcal{P}$ is a space-like plane that is $\forall \overrightarrow{u} \in \overrightarrow{P}$, $\overrightarrow{u}^2 > 0$ then $\mathcal{C}$ is drawn as an ellipse(Fig.3.(a)) . The set $\mathcal{C}$ represents a based circle sphere pencil where all spheres get a common circle.
- If $\mathcal{P}$ is a light-like plane that is $\forall \overrightarrow{u} \in \overrightarrow{P}$, $\overrightarrow{u}^2 = 0$ and $\mathcal{P}$ is parallel to a hyperplane tangent at $\mathcal{C}_l$. (Fig.3.(b)) Then the set $\mathcal{C}$ is drawn as two straight lines symmetric wrt O_5. All spheres in the pencil are tangent at a point.
- If $\mathcal{P}$ is a time-like plane that is $\forall \overrightarrow{u} \in \overrightarrow{P}$, $\overrightarrow{u}^2 < 0$ then $\mathcal{C}$ is drawn as a hyperbola and forms a limited points pencil. (Fig.3.(c))These points are obtained from the light directions of $\mathcal{P}$.

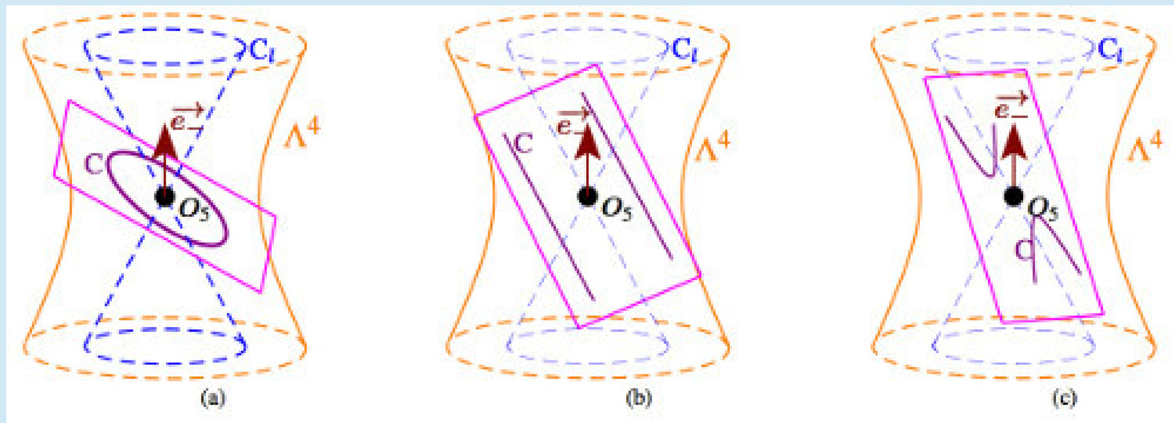

Figure 2: The representation of the three spheres pencil types on Λ^4

Canal surfaces on Λ^4

The envelop of a one-parameter set of oriented spheres in $\mathbb{R}^3$ defines a canal surface. The cones and the Dupin cyclides are known examples of canal surfaces of degree 2. On Λ^4, any curve $t \to \sigma(t)$ represents a canal surface. Its characteristic circles are obtained by the intersection of 2 particular spheres (Fig 4).

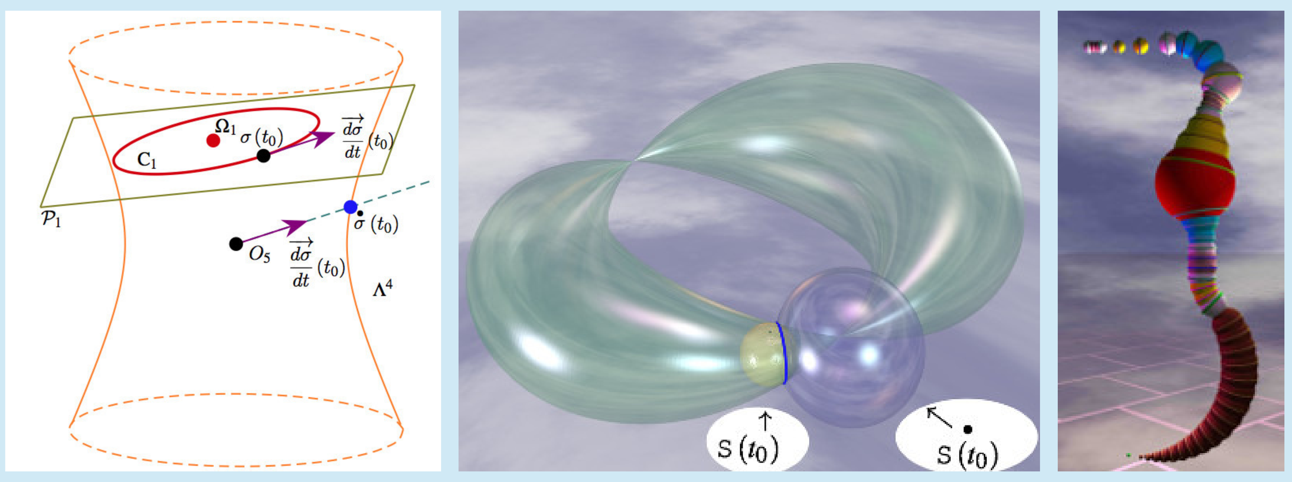

Figure 3: A Dupin cyclide on Λ^4 (left) on $\mathbb{R}^3$ (centre) and seahorse (right)

On Λ^4 the circle C_1 represents a Dupin cyclide. The tangent vector at the curve on point $\sigma(t_0)$ is given by $\dfrac{d\overrightarrow{\sigma}}{dt}(t_0)$. The characteristic circle of the Dupin cyclide is provided by the intersection of the two spheres $S(t_0)$ and $\overset{\bullet}{S}(t_0)$. These spheres are represented in Λ^4 by $\sigma(t_0)$ and $\overset{\bullet}{\sigma}(t_0)$.The last sphere is obtained by the intersection between the half line $\left[O_5; \dfrac{d\overrightarrow{\sigma}}{dt}(t_0)\right)$ and Λ^4. The Figure 5 shows two cyclides from Λ^4 to $\mathbb{R}^3$.

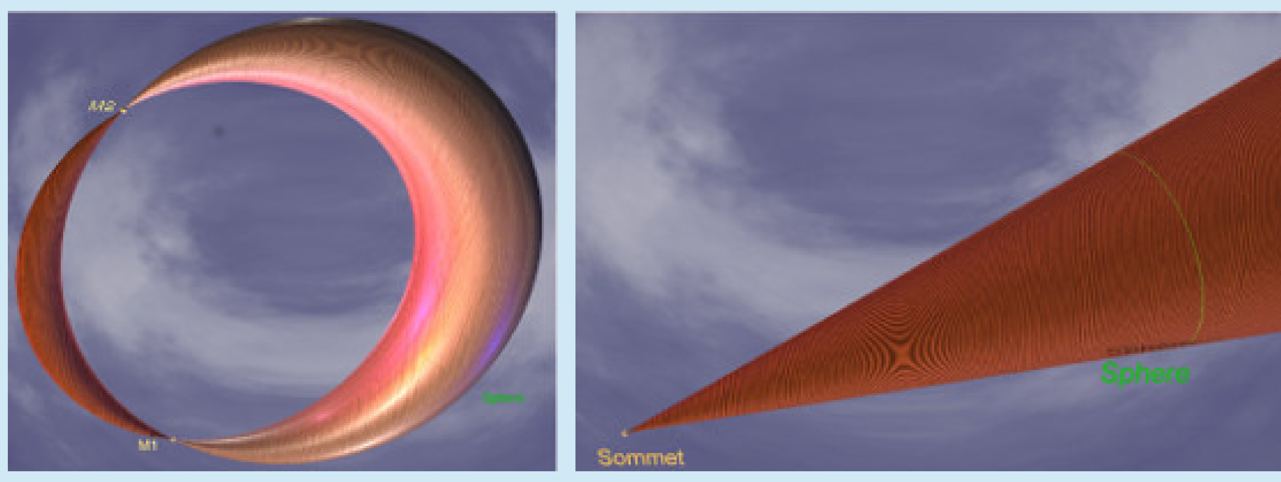

Figure 4: The same representation, on Λ^4 of a Dupin cyclide and of a circular cone: the implementation is the same with or without the point at infinity of $\mathbb{R}^3$ (M_2 is send to the infinity), the modeling is the same as envelope of spheres or planes (Dupin cyclide or circular cone).

Conclusion : The Minkowski-Lorentz space offers a new way to handle curves and surfaces for CAD purposes making the computation easier. Algorithms for G1 joins, not given here, are used to sketch a seahorse as example.

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