Fuzzy mortality models based on orthonormal expansion of membership functions

Andrzej Szymański and Agnieszka Rossa, Institute of Statistics and Demography
University of Lodz, POLAND

University of LODZ

Aim and scope

The new trends in fuzzy analysis are based on the algebraic approach to fuzzy numbers. The essential idea is representing the membership function of a fuzzy number as an element of any square-integrable function space. Koissi and Shapiro (2006) proposed symmetric triangular membership functions to define fuzzy numbers used in their fuzzy version of the well-known Lee–Carter mortality model. In our approach a mortality model is proposed based on orthonormal expansions of exponential membership functions.

Notation

Let $m_x(t)$ denote an age-specific crude mortality rate

$$m_x(t) = \frac{D_x(t)}{\bar{N}_x(t)}, \tag{1}$$

where:

- $x = 0, 1, \ldots, \omega$, $t = 1, 2, \ldots, T$ – indices denoting age groups $[x, x+1)$ and calendar years, respectively,
- $D_x(t)$ – number of deaths at age x in year t,
- $\bar{N}_x(t)$ – midyear population size at age x.

Lee-Carter model LC (Lee, Carter 1992)

$$\ln m_x(t) = a_x + b_x k_t + \xi_{xt}, \tag{2}$$

under constraints

$$\sum_{t=1}^{T} k_t = 0, \; \sum_{x=0}^{\omega} b_x = 1, \tag{3}$$

- m_{xt} – crude death rates defined in (1),
- k_t – time parameters indexed by t,
- a_x, b_x – age-specific parameters indexed by x,
- ξ_{xt} – independent random errors, $\xi_{xt} \sim N(0, \sigma_\xi^2)$.

Fuzzy version of the LC model (Koissi, Shapiro 2006)

$$Y_{xt} = A_x \oplus (B_x \otimes K_t), \tag{4}$$

where:

- $Y_{xt} = (\ln m_{x,t}, e_{x,t})$ – symmetric triangular numbers representing fuzzy log-central death rates,
- $A_x = (a_x, s_{A_x})$, $B_x = (b_x, s_{B_x})$, $K_t = (k_t, s_{K_t})$ – symmetric triangular fuzzy numbers with unknown central values a_x, b_x, k_t and spreads $s_{A_x}, s_{B_x}, s_{K_t}$.

Decomposition of a monotonic membership function

Suppose that the membership function $\mu(z)$ of a fuzzy number is strictly monotonic on two disjoint intervals. We can decompose $\mu(z)$ into two parts: strictly increasing and strictly decreasing functions $\Psi(z)$ and $\Phi(z)$. Let us consider a membership function of the form

$$\mu(z) = \begin{cases} \exp\{-\left(\frac{c-z}{\tau}\right)^2\} & \text{for } z \le c, \\ \exp\{-\left(\frac{z-c}{\nu}\right)^2\} & \text{for } z > c. \end{cases} \tag{5}$$

Exponential membership function

Let us denote

$$\begin{aligned} \Psi(z) &= \exp\left\{-\left(\frac{c-z}{\tau}\right)^2\right\} \quad \text{for } z \le c, \\ \Phi(z) &= \exp\left\{-\left(\frac{z-c}{\nu}\right)^2\right\} \quad \text{for } z > c, \end{aligned} \tag{6}$$

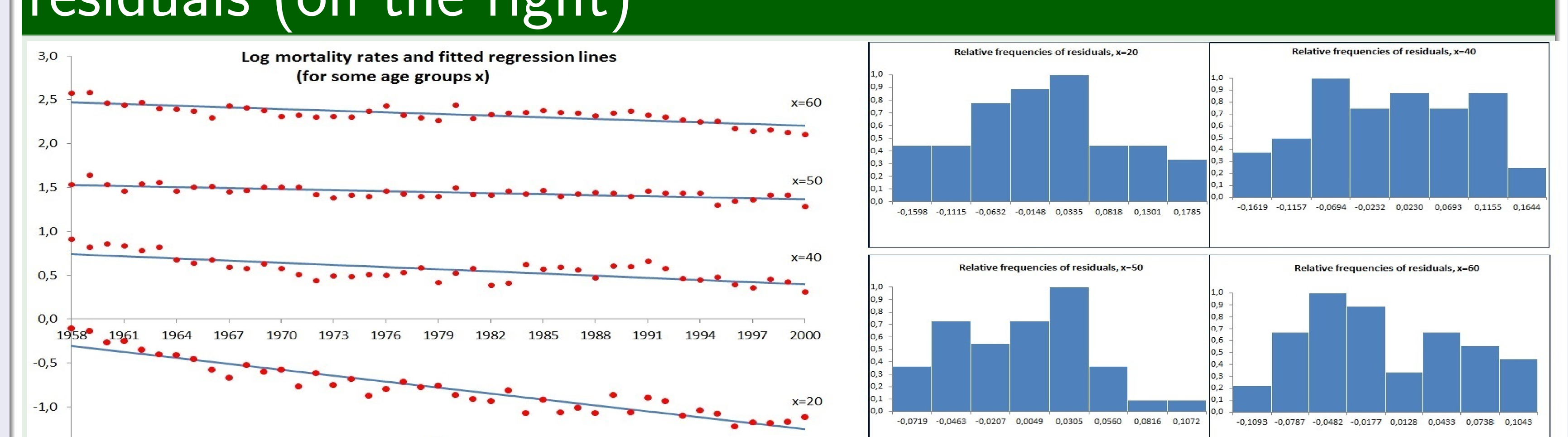
then there exist inverse functions $\Psi^{-1}(z)$ and $\Phi^{-1}(z)$.

Inverse exponential membership functions

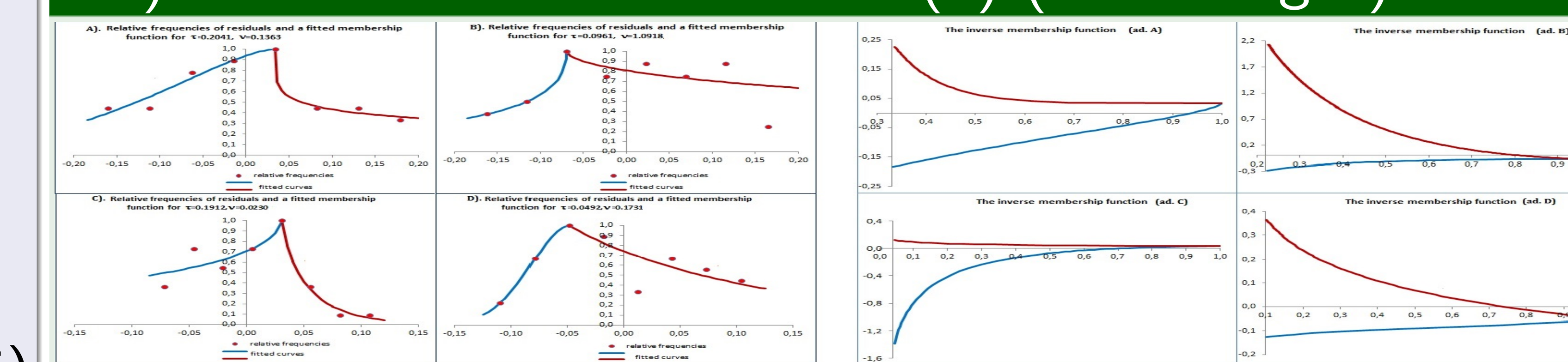
$$\begin{aligned} f(u) &= \Psi^{-1}(u) = c + \psi(u) = c - \tau(-\ln u)^{-\frac{1}{2}} \\ g(u) &= \Phi^{-1}(u) = c + \phi(u) = c + \nu(-\ln u)^{-\frac{1}{2}} \end{aligned} \tag{7}$$

and $f, g \in L^2(0,1)$. Parameters c, τ, ν are found using the Nasibov–Peker (2011) approach (see Example).

Example. Log-central mortality rates and fitted regression lines (on the left) for some x; relative frequencies of residuals (on the right)



Example cont. Exponential membership functions (5) adjusted to the relative frequencies of residuals (on the left) and their inverse functions (7) (on the right)



Expressing the inverse exponential membership functions by means of orthonormal expansions in $L^2(0,1)$

Let us consider inverse exponential membership functions (7). For an orthonormal set of vectors P_j and for any vector $f \in L^2(0,1)$ the following expansion holds

$$f = \sum_{j=0}^{\infty} \langle P_j, f \rangle P_j \text{ and } \langle P_j, f \rangle = \int_0^1 P_j(u) f(u) du. \tag{8}$$

We suggest using Legendre's polynomials $P_0, \ldots, P_4$ in (8) as a good approximation of f.

Mortality model based on exponential fuzzy numbers

We propose a new mortality model based on (4) as

$$\tilde{Y}_{x,t} = \tilde{A}_x + b_x \tilde{K}_t, \quad x = 1, \ldots, \omega, t = 1, \ldots, T, \tag{9}$$

where:

- $\tilde{Y}_{x,t} \in \mathrm{R}^n$ – vector of expansion coefficients of inverse exponential function (7) with parameters $c_{x,t} = \ln m_{x,t}$, $\tau = \tau_x$, $\nu = \nu_x$ which are obtained by means of the Nasibov–Peker method,
- b_x – some unknown scalar parameters,
- $\tilde{A}_x, \tilde{K}_t \in \mathrm{R}^n$ – vectors of expansion coefficients of inverse exponential functions (7) with unknown parameters $c = a_x, b_x, k_t$, $\tau = \tau_{A_x}, \tau_{B_x}, \tau_{K_t}$ and $\nu = \nu_{A_x}, \nu_{B_x}, \nu_{K_t}$, respectively, subject to the weighted least squares estimation.

References

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